Mixed Intermediate Jacobians

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1 Introduction

The main purpose of this thesis is to describe the structure of intermediate Jacobians of the product of two Kähler manifolds. Given two Kähler manifolds X, Y we construct a decomposition of the k-th intermediate Jacobian of $X \times Y$ as a product of mixed intermediate Jacobians of X and Y.

$$J^{2k-1}(X \times Y) = \prod_{l+m=2k-1} J^{l,m}(X,Y)$$

We study how this decomposition behaves with respect to the Abel-Jacobi map and the duality between $J^{2k-1}(X \times Y)$ and $J^{2(n_1+n_2)-(2k-1)}(X \times Y)$ where n_1, n_2 are the dimensions of X and Y respectively. Given a positive integer k and an analytic cycle U in Y of codimension l, there exists a homomorphism of tori

$$\Psi_U^k : J^{2k-1}(X) \to J^{2k-1,2l}(X,Y)$$

such that for any analytic cycle Z in X of codimension k homologous to 0 we have

$$\Psi_U^k \circ \Phi_X^k(Z) = \Phi_{X \times Y}^{k+l}(Z \times U)$$

where Φ_X^k and $\Phi_{X \times Y}^{k+l}$ are the Abel-Jacobi maps. We also have that the duality between $J^{2k-1}(X \times Y)$ and $J^{2(n_1+n_2)-(2k-1)}(X \times Y)$ induced by the Poincare duality on the cohomology groups induces a duality between $J^{l,m}(X,Y)$ and $J^{2n_1-l,2n_2-m}(X,Y)$. In the last section we give an example of when a product of Kähler manifolds naturally arises as the Jacobian of a degenerate fiber of a family whose generic fiber is a Riemann surface.

In Sections 2 and 3 we give a brief introduction to Kähler manifolds, intermediate Jacobians and Appell-Humbert theory of line bundles on complex tori. The main references for Sections 2 and 3 are [5] and [1] respectively. All of the original work is concentrated in Sections 4 and 5. We will assume basic knowledge of complex manifolds, algebraic topology and sheaf theory.

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2 Kähler Manifolds and Intermediate Jacobians

2.1 Complex Manifolds

Let X be a complex manifold. Let $TX_{\mathbb{R}}$ be the tangent bundle of X considered as a real differentiable vector bundle and let $TX_{\mathbb{C}}$ be its complexification

$$TX_{\mathbb{C}} := TX_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

Let I be the complex structure on $TX_{\mathbb{R}}$, that is the endomorphism of $TX_{\mathbb{R}}$ given by multiplication by i in TX, the tangent bundle of X considered as a complex vector bundle. We have the decomposition of $TX_{\mathbb{C}}$ into the eigenspaces of I,

$$TX_{\mathbb{C}} = T^{1,0}(X) \oplus T^{0,1}(X)$$

where $T^{1,0}(X)$ is the eigenspace corresponding to the eigenvalue i and $T^{0,1}(X)$ is the eigenspace corresponding to the eigenvalue $-i$. This induces the decomposition of the complexified cotangent bundle

$$\Omega(X) := TX_{\mathbb{C}}{}^* = \Omega^{1,0}(X) \oplus \Omega^{0,1}(X)$$

where $\Omega^{1,0}(X)$ is the bundle of $\mathbb{C}$-linear 1-forms on X and $\Omega^{0,1}(X)$ is the bundle of $\mathbb{C}$-antilinear 1-forms on X. The bundle $T^{1,0}(X)$ is naturally isomorphic to the complex tangent bundle TX and thus inherits a holomorphic structure. There is a natural complex conjugation on the bundle $TX_{\mathbb{C}}$ given by complex conjugation on $TX_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ which also induces complex conjugation on $\Omega(X)$. It is easy to check that

$$\begin{aligned} \overline{T^{1,0}(X)} &= T^{0,1}(X), \\ \overline{\Omega^{1,0}(X)} &= \Omega^{0,1}(X). \end{aligned}$$

We also have a decomposition of differential k-forms for any non-negative integer k,

$$\Omega^k(X) = \bigoplus_{i+j=k} \Omega^{i,j}(X)$$

where $\Omega^{i,j}(X) = \left(\bigwedge^i \Omega^{1,0}(X)\right) \otimes \left(\bigwedge^j \Omega^{0,1}(X)\right)$. We say that a differential k-form is of type (i,j) if it is a section of $\Omega^{i,j}(X)$. Let A^k,$A^{i,j}$ be the sheaves of differentiable sections of $\Omega^k(X)$ and $\Omega^{i,j}(X)$ respectively. For any $\alpha \in A^{i,j}(X)$ we have that

$$d\alpha = (d\alpha)^{i+1,j} + (d\alpha)^{i,j+1}$$

where $(d\alpha)^{i+1,j} \in A^{i+1,j}(X)$ and $(d\alpha)^{i,j+1} \in A^{i,j+1}(X)$. We define the operators ∂ and $\overline{\partial}$ by

$$\begin{aligned} \partial\alpha : &= (d\alpha)^{i+1,j}, \\ \overline{\partial}\alpha : &= (d\alpha)^{i,j+1}. \end{aligned}$$

We extend ∂ and $\overline{\partial}$ by linearity to the entire space of differentiable k-forms $A^k(X)$ for every k.

2.2 Kähler Manifolds

Given a complex vector space V, let $V_{\mathbb{R}}$ be the real vector space where we forget the complex structure on V and let $V_{\mathbb{C}}$ be the complexification of $V_{\mathbb{R}}$

$$V_{\mathbb{C}} = V_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}.$$

If we have a hermitian form $h : V \times V \to \mathbb{C}$, we can consider the two-form on $V_{\mathbb{R}}$ given by

$$\omega := -\Im h.$$

Moreover, if we extend ω by linearity to $V_{\mathbb{C}}$, we have that ω is of type $(1,1)$. We have that the symmetric form $g := \Re h$ on $V_{\mathbb{R}}$ is given by

$$g(u,v) = \omega(u, Iv)$$

where I is the complex structure on $V_{\mathbb{R}}$. In fact, there is a one-to-one correspondence between real $(1,1)$ forms on $V_{\mathbb{C}}$ and the hermitian forms on V given by

$$\omega \mapsto h = g - i\omega.$$

A hermitian metric h on a complex manifold X is a differentiable section of $(TX^* \otimes \overline{TX^*})$ such that on each fiber of TX it is a positive definite hermitian form. From the discussion above, we see that given a hermitian metric h on X, the real two-form on $TX_{\mathbb{R}}$ given by

$$\omega := -\Im h$$

is of type $(1,1)$ when extended to $TX_{\mathbb{C}}$ by linearity. We also associate to h the symmetric 2-form $g := \Re h$. Clearly both g and ω are non-degenerate.

Definition 2.2.1. A Kähler metric on a complex manifold X is a hermitian metric h on X, such that the corresponding real two-form ω is closed. In this case ω is called the Kähler form corresponding to h.

We say that a complex manifold is Kähler if it admits a Kähler metric. It will become more evident later on why Kähler manifolds are important and what good properties they possess, but for now we give an indication for why they might be interesting to study.

Theorem 2.2.2. *Let X be a complex manifold of dimension n with a hermitian metric h. The metric h is Kähler if and only if for every $x \in X$, there exist local holomorphic coordinates $z_1, \ldots, z_n$ around x such that the matrix of h with respect to these coordinates is given by*

$$h = I_n + O(\sum_{i=1}^{n} |z_i|^2)$$

where I_n is the identity matrix.

In other words, Kähler manifolds are those that admit a metric that is locally constant to the first order.

Example 2.2.3. We will construct the Fubini-Study metric on a projective space $\mathbb{P}^n$ and thus show that $\mathbb{P}^n$ is a Kähler manifold. We first introduce the Chern form of a holomorphic line bundle with a hermitian metric. Let X be a complex manifold and let L be a holomorphic line bundle on X endowed with a hermitian metric $\mathfrak{h}$. Let $\{U_i\}_{i \in I}$ be a cover of X which trivializes L and let $\sigma_i \in \Gamma(L, U_i)$ be non-vanishing holomorphic sections of L over the open sets U_i corresponding to the constant section equal to 1 in the trivializations. We have that the transition functions $g_{ij} \colon U_i \cap U_j \to \mathbb{C}$ are given by

$$\sigma_i = g_{ij} \cdot \sigma_j.$$

Consider the functions $\mathfrak{h}_i = \mathfrak{h}(\sigma_i, \sigma_i)$ defined on U_i and the two-forms

$$\omega_i = \frac{1}{2\pi i} \partial\overline{\partial} \log \mathfrak{h}_i.$$

We have $\mathfrak{h}_i = |g_{ij}|^2 \mathfrak{h}_j$ on $U_i \cap U_j$ and $\partial\overline{\partial} \log |g_{ij}|^2 = \partial\overline{\partial} \log g_{ij} + \partial\overline{\partial} \log \overline{g_{ij}} = 0$ since the functions g_{ij} are holomorphic. In particular ω_i coincide on the intersections $U_i \cap U_j$ and therefore define a global two-form ω. It is clear from the construction that ω is real, closed and of type $(1,1)$.

Let $\mathcal{O}_{\mathbb{P}^n}(1)$ be the dual of the tautological line bundle S over $\mathbb{P}^n$. We have that S is a sub-bundle of $\mathbb{C}^{n+1} \times \mathbb{P}^n$. Let $\mathfrak{h}$ be the restriction of the standard hermitian metric on $\mathbb{C}^{n+1}$ to S. We have that $\mathfrak{h}$ induces a metric on $\mathcal{O}_{\mathbb{P}^n}(1)$ which we will denote by $\mathfrak{h}^*$. Let ω be the Chern form corresponding to $\mathcal{O}_{\mathbb{P}^n}(1)$ and $\mathfrak{h}^*$. The hermitian form h on $\mathbb{P}^n$ corresponding to ω is in fact positive definite and therefore Kähler. The metric h is called the Fubini-Study metric.

Example 2.2.4. It is easy to see that the restriction of a Kähler metric to a complex submanifold is Kähler and therefore a complex submanifold of a Kähler manifold is Kähler. Along with the previous example, this shows that projective complex manifolds are Kähler.

2.3 Hodge Decomposition

Hodge decomposition is a crucial tool in studying complex manifolds and their cohomology groups. Since in general cohomology groups are represented by quotients of infinite dimensional spaces, it is sometimes difficult to understand their precise structure. In the case of a compact manifold, the Hodge theory provides a concrete way of representing some cohomology groups as objects with some analytic properties.

Let X be a compact complex manifold of dimension n and let g be a Riemannian metric on $TX_{\mathbb{R}}$. This metric defines a hermitian metric on $\Omega^k(X)$ for all k.

Definition 2.3.1. The Hodge operator $* : A^k(X) \to A^{2n-k}(X)$ is defined by

$$(\alpha_x, \beta_x)\mathrm{Vol}_x = \alpha_x \wedge \overline{*\beta_x}$$

where Vol_x is the volume form at x defined by the metric g and this equality holds for every $x \in X$.

The Hodge operator satisfies the following identity on $A^k(X)$:

$$*^2 = (-1)^k.$$

Definition 2.3.2. We define the operator $d^* = - * d*$ on $A^k(X)$. It is the adjoint of d with respect to the L^2 metric on $A^k(X)$

$$(\alpha, \beta)_{L^2} = \int_X (\alpha_x, \overline{\beta_x})\mathrm{Vol}_x$$

which exists since the manifold is compact.

Definition 2.3.3. The laplacian operator is defined by

$$\Delta_d = dd^* + d^*d$$

Definition 2.3.4. A k-form $\alpha \in A^k(X)$ is called harmonic if $\Delta_d(\alpha) = 0$. The space of harmonic k-forms is denoted by $\mathcal{H}^k(X, \mathbb{C})$.

One can show that the harmonic forms are exactly those forms lying in $\ker d \cap \ker d^*$. In particular, for any non-negative integer k, there is a map $\mathcal{H}^k(X, \mathbb{C}) \to H^k_{\mathrm{dR}}(X, \mathbb{C})$ which sends a harmonic form to its class in the de Rham cohomology of X with coefficients in $\mathbb{C}$. Using the theory of elliptic operators, one deduces that this map is an isomorphism. In particular we have

$$\mathcal{H}^k(X, \mathbb{C}) \cong H^k_{\mathrm{dR}}(X, \mathbb{C}).$$

Using this concrete representation of de Rham cohomology groups, we can show that the pairing $H^p_{\mathrm{dR}}(X, \mathbb{C}) \otimes H^{2n-p}_{\mathrm{dR}}(X, \mathbb{C}) \to \mathbb{C}$ given by

$$(\alpha, \beta) = \int_X \alpha \wedge \beta$$

is perfect for every $p \in [0, 2n]$.

In the case when our compact complex manifold is Kähler, we can introduce even more structure to its cohomology groups. Let X be a compact Kähler manifold with a Kähler metric h, Kähler form ω and the corresponding Riemannian metric g. Let $\partial^* := - * \overline{\partial} *$ and $\overline{\partial}^* := - * \partial *$ be the adjoints of ∂ and $\overline{\partial}$ with respect to the metric $(\cdot, \cdot)_{L^2}$ on $A^k(X)$. The laplacian operators corresponding to ∂ and $\overline{\partial}$ are defined by

$$\Delta_\partial = \partial\partial^* + \partial^*\partial, \qquad \Delta_{\overline{\partial}} = \overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial}.$$

In the Kähler case, we have the equality

$$\Delta_\partial = \Delta_{\overline{\partial}} = \frac{1}{2}\Delta_d.$$

We have that Δ_∂ preserves the types of forms and therefore in this case, so does Δ_d,

$$\Delta_d(A^{p,q}(X)) \subset A^{p,q}(X).$$

If $\alpha \in \mathcal{H}^k(X, \mathbb{C})$ is a harmonic form, and $\alpha = \sum_{i+j=k} \alpha^{i,j}$ is its decomposition into forms of type (i, j), then we must have that $\alpha^{i,j}$ are also harmonic for every i, j. This gives us the decomposition for every k,

$$\mathcal{H}^k(X, \mathbb{C}) = \bigoplus_{i+j=k} \mathcal{H}^{i,j},$$

where $\mathcal{H}^{i,j}$ is the space of harmonic forms of type (i, j). The isomorphism $\mathcal{H}^k(X, \mathbb{C}) \cong H^k_{\mathrm{dR}}(X, \mathbb{C})$ gives us the corresponding decomposition of the de Rham cohomologies

$$H^k_{\mathrm{dR}}(X, \mathbb{C}) = \bigoplus_{i+j=k} H^{i,j}_{\mathrm{dR}}(X, \mathbb{C}).$$

In fact one can show that $H^{i,j}_{dR}(X,\mathbb{C})$ consists of classes of closed forms which are representable by forms of type (i,j), and thus in particular this decomposition does not depend on the choice of a Kähler metric on X. We also have that the pairing $H^{i,j}_{\mathrm{dR}}(X,\mathbb{C}) \otimes H^{n-i,n-j}_{\mathrm{dR}}(X,\mathbb{C}) \to \mathbb{C}$ given by

$$(\alpha,\beta) = \int_X \alpha \wedge \beta$$

is perfect.

By the theorem of de Rham, for every non-negative integer k, we have a canonical isomorphism

$$H^k_{\mathrm{dR}}(X,\mathbb{C}) \cong H^k(X,\mathbb{C})$$

where $H^k(X,\mathbb{C})$ is the k-th singular cohomology group of X with coefficients in $\mathbb{C}$. We thus also have the Hodge decomposition of singular cohomologies with coefficients in $\mathbb{C}$

$$H^k(X,\mathbb{C}) = \bigoplus_{i+j=k} H^{i,j}(X).$$

Let Ω^k_X be the sheaf of holomorphic sections of $\Omega^{k,0}(X)$. We can calculate the i-th sheaf cohomology of Ω^k_X using the exact Dolbeault sequence of sheaves

$$0 \longrightarrow \Omega^k_X \longrightarrow A^{k,0} \xrightarrow{\overline{\partial}} A^{k,1} \xrightarrow{\overline{\partial}} \cdots .$$

Since the harmonic forms are $\overline{\partial}$-closed we have a map

$$\mathcal{H}^{k,i} \to H^i(X,\Omega^k_X)$$

which sends a harmonic form α of type (k,i) to its class in

$$H^i(X,\Omega^k_X) \cong \frac{\ker(\overline{\partial} : A^{k,i}(X) \to A^{k,i+1}(X)}{\mathrm{Im}(\overline{\partial} : A^{k,i-1}(X) \to A^{k,i}(X)}$$

which is in fact an isomorphism. Moreover, the map induced on the Hodge components of singular cohomology

$$H^{k,i}(X) \xrightarrow{\sim} H^i(X,\Omega^k_X)$$

does not depend on the choice of the Kähler metric on X.

2.4 Hodge Structures

Definition 2.4.1. An integral Hodge structure of weight k where k is a non-negative integer is a pair $(V_{\mathbb{Z}}, \{V^{p,q}\}_{p,q\geq 0, p+q=k})$ where $V_{\mathbb{Z}}$ is a free abelian group of finite rank and $\{V^{p,q}\}$ gives a decomposition

$$V_{\mathbb{C}} := V_{\mathbb{Z}} \otimes \mathbb{C} = \bigoplus_{p+q=k} V^{p,q}$$

such that $\overline{V^{p,q}} = V^{q,p}$. The Hodge structure is also denoted by $(V_{\mathbb{Z}}, V^{p,q})$.

A Hodge structure $(V_{\mathbb{Z}}, V^{p,q})$ of weight k defines a filtration on the vector space $V_{\mathbb{C}}$ by

$$F^l V_{\mathbb{C}} := \bigoplus_{p\geq l} V^{p,k-p}.$$

This filtration determines the Hodge structure on $V_{\mathbb{Z}}$ since we have

$$V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}.$$

Definition 2.4.2. Let $V = (V_{\mathbb{Z}}, V^{p,q})$ and $W = (W_{\mathbb{Z}}, W^{p,q})$ be integral Hodge structures of weight k. We define the direct sum $(V \oplus W)$ of Hodge structures in the following way,

$$\begin{aligned} (V\oplus W)_{\mathbb{Z}} &= V_{\mathbb{Z}} \oplus W_{\mathbb{Z}}, \\ (V\oplus W)^{p,q} &= V^{p,q} \oplus W^{p,q}. \end{aligned}$$

Definition 2.4.3. Let $V = (V_{\mathbb{Z}}, V^{p,q})$ and $W = (W_{\mathbb{Z}}, W^{p,q})$ be two integral Hodge structures of weights k and l respectively. We define the tensor product of V and W as the Hodge structure of weight $k + l$ given by

$$\begin{aligned} (V \otimes W)_{\mathbb{Z}} &= V_{\mathbb{Z}} \otimes W_{\mathbb{Z}}, \\ (V \otimes W)^{r,s} &= \bigoplus_{p+p'=r, q+q'=s} V^{p,q} \otimes W^{p',q'}. \end{aligned}$$

Let X be a compact Kähler manifold. By the previous section, for any integer k we have the Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{i+j=k} H^{i,j}(X).$$

On the level of singular cohomologies we have by the universal coefficient theorem

$$H^k(X, \mathbb{C}) = H^k(X, \mathbb{Z}) \otimes \mathbb{C}.$$

This defines a complex conjugation on the vector space $H^k(X, \mathbb{C})$. We have that under the isomorphism $H^k(X, \mathbb{C}) \cong \mathcal{H}^k(X)$, this complex conjugation coincides with the complex conjugation on $\mathcal{H}^k(X)$ inherited from the complex conjugation on $\Omega^k(X)$. In particular we have that

$$\overline{H^{i,j}(X)} = H^{j,i}(X).$$

The Hodge decomposition of $H^k(X, \mathbb{C})$ thus defines an integral Hodge structure of weight k,

$$H^k(X) = (H^k(X, \mathbb{Z})_0, H^{i,j}(X))$$

and the Hodge filtration

$$F^l H^k(X, \mathbb{C}) = \bigoplus_{i \geq l} H^{i,k-i}(X).$$

Here and everywhere in what follows, by $H^k(X, \mathbb{Z})_0$ we mean $H^k(X, \mathbb{Z})/\text{torsion}$.

Let X, Y be two compact Kähler manifolds. For every non-negative k, the Künneth formula provides an isomorphism

$$H^k(X \times Y, \mathbb{Z})_0 \cong \bigoplus_{p+q=k} H^p(X, \mathbb{Z})_0 \otimes H^q(Y, \mathbb{Z})_0$$

given by the cup product of cocycles. On the level of de Rham cohomologies, cup product is given by the wedge product of forms and therefore preserves the Hodge decomposition

$$H^{r,s}(X \times Y) \cong \bigoplus_{p+p'=r, q+q'=s} H^{p,q}(X) \otimes H^{p',q'}(Y).$$

In particular this means that for any integer k, we have an isomorphism of Hodge structures

$$H^k(X \times Y) \cong \bigoplus_{p+q=k} H^p(X) \otimes H^q(Y).$$

2.5 Analytic Cycles

Analytic cycles will play an important role in understanding the objects we will be considering in the following chapters.

Definition 2.5.1. A closed subset Z of a complex manifold X is called an analytic set if there exists an open cover $\{U_i\}_{i \in I}$ of X such that for all $i \in I$, there exist holomorphic functions $f_1, \ldots, f_N$ on U_i such that $Z \cap U_i$ is the zero set of these functions.

Even though in general, analytic sets are not smooth, we have the following theorem that makes them more approachable.

Theorem 2.5.2. *Let $Z \subset X$ be an analytic set. There exists a nowhere dense analytic subset $Z_{\text{sing}} \subset Z$ such that $Z_{\text{smooth}} := Z \backslash Z_{\text{sing}}$ is a complex submanifold of X.*

Definition 2.5.3. An analytic set $Z \subset X$ is called irreducible if Z_{smooth} is connected. In that case the dimension of Z is defined as the complex dimension of Z_{smooth}.

Definition 2.5.4. An analytic cycle of dimension k is a finite combination with integer coefficients of irreducible analytic sets of dimension k.

Let X be a compact complex manifold. We have that irreducible analytic sets can be finitely triangulated by differentiable chains and thus to every k-dimensional analytic cycle U in X corresponds an element in $H_{2k}(X, \mathbb{Z})$ which we will denote by $\langle U \rangle$. The cohomology class of an analytic set U is defined by

$$[U] := P(\langle U \rangle) \in H^{2n-2k}(X, \mathbb{Z})$$

where

$$P : H_{2k}(X, \mathbb{Z}) \xrightarrow{\sim} H^{2n-2k}(X, \mathbb{Z})$$

is the Poincaré duality map.

For an irreducible analytic set Z of dimension k and a closed $2k$-form α we have

$$[Z] \wedge [\alpha] = \int_{Z_{\text{smooth}}} \alpha$$

where $[\alpha] \in H^{2k}_{\text{dR}}(X, \mathbb{C})$ is the class of α and where we identify $H^{2n}_{\text{dR}}(X, \mathbb{C})$ with $\mathbb{C}$ via integration of closed forms over X. From this we can see that if X is a compact Kähler manifold of dimension n, then for any k-dimensional analytic cycle U in X, we have

$$[U] \in H^{n-k,n-k}(X).$$

2.6 Intermediate Jacobians

Let X be a compact Kähler manifold of dimension n. For any positive integer k, we have

$$H^{2k-1}(X, \mathbb{C}) = F^k H^{2k-1}(X) \oplus \overline{F^k H^{2k-1}(X)}.$$

It implies that $F^k H^{2k-1}(X) \cap H^{2k-1}(X, \mathbb{R}) = \{0\}$ and the map

$$\phi \colon H^{2k-1}(X, \mathbb{R}) \to H^{2k-1}(X, \mathbb{C})/F^k H^{2k-1}(X)$$

is an isomorphism of real vector spaces. The image of $H^{2k-1}(X, \mathbb{Z})$ under the map ϕ is therefore a lattice of full rank in $H^{2k-1}(X, \mathbb{C})/F^k H^{2k-1}(X)$.

Definition 2.6.1. The k-th intermediate Jacobian of X is the complex torus

$$J^{2k-1}(X) = \left(H^{2k-1}(X, \mathbb{C})/F^k H^{2k-1}(X)\right) / \phi(H^{2k-1}(X, \mathbb{Z})).$$

For any non-negative integers k and l, the cup product gives an isomorphism

$$H^{k,l}(X)^* \cong H^{n-k,n-l}(X)$$

once we identify $H^{n,n}(X) = H^{2n}(X, \mathbb{C})$ with $\mathbb{C}$ via integration of closed forms over X. We also have that the Poincaré duality gives an isomorphism

$$H^{2k-1}(X, \mathbb{Z}) \cong H_{2n-2k+1}(X, \mathbb{Z}).$$

We can thus realize the k-th intermediate Jacobian of X as

$$J^{2k-1}(X) \cong (F^{n-k+1} H^{2n-2k+1}(X, \mathbb{C}))^* / H_{2n-2k+1}(X, \mathbb{Z})$$

where $H_{2n-2k+1}(X, \mathbb{Z})$ acts on $F^{n-k+1} H^{2n-2k+1}(X, \mathbb{C})$ by integration over differentiable cycles.

We will now define the Abel-Jacobi map Φ^k_X from the group $\mathcal{Z}^k(X)_{\text{hom}}$ of analytic cycles of codimension k homologous to 0 to the k-th intermediate Jacobian of X. Let $Z \in \mathcal{Z}^k(X)_{\text{hom}}$. Since Z is homologically trivial, we can triangulate Z and find a differentiable chain Γ of real dimension $2n-2k+1$ such that $\partial \Gamma = Z$. One can show that

$$(F^{n-k+1} H^{2n-2k+1}(X, \mathbb{C})) \cong \frac{F^{n-k+1} A^{2n-2k+1}(X) \cap \ker(d)}{d F^{n-k+1} A^{2n-2k}(X)},$$

where $F^{n-k+1}A^{2n-2k+1}(X) = \bigoplus_{i\geq n-k+1} A^{i,2n-2k+1-i}(X)$. In other words, if $\alpha, \beta \in F^{n-k+1}A^{2n-2k+1}(X)$ are closed and define the same class in $H^{2n-2k+1}(X,\mathbb{C})$, then there exists a form $\gamma \in F^{n-k+1}A^{2n-2k}(X)$ such that $\alpha - \beta = d\gamma$. For $\alpha \in (F^{n-k+1}H^{2n-2k+1}(X,\mathbb{C}))$, we define $\int_\Gamma \alpha$ in the following way: pick a representative $\beta \in F^{n-k+1}A^{2n-2k+1}(X)$ for α and define

$$\int_\Gamma \alpha = \int_\Gamma \beta.$$

The choice of representative of α does not change the result since it would differ by

$$\int_\Gamma d\phi = \int_{Z_{\text{smooth}}} \phi$$

where $\phi \in F^{n-k+1}A^{2n-2k}(X)$ which is 0 since $F^{n-k+1}A^{2n-2k}(Z_{\text{smooth}}) = 0$ due to type. We thus have that $\int_\Gamma$ defines an element in $(F^{n-k+1}H^{2n-2k+1}(X,\mathbb{C}))^*$. If we pick a different Γ' such that $\partial\Gamma' = Z$, we have that $\Gamma - \Gamma' \in H_{2n-2k+1}(X,\mathbb{Z})$, and therefore Γ and Γ' define the same element in $J^{2k-1}(X)$. This construction thus defines the desired map.

When defining the action of $\int_\Gamma$ on $F^{n-k+1}H^{2n-2k+1}(X,\mathbb{C})$, we could have fixed a Kähler metric on X and for $\alpha \in F^{n-k+1}H^{2n-2k+1}(X,\mathbb{C})$ we could have defined

$$\int_\Gamma \alpha := \int_\Gamma \tilde{\alpha}$$

where $\tilde{\alpha}$ is the harmonic $2n-2k+1$-form representing α. The argument above shows that this definition would not depend on the choice of Kähler metric.

Example 2.6.2. The first intermediate Jacobian of X has a more familiar form. We have

$$J^1(X) = H^{0,1}(X)/H^1(X,\mathbb{Z}).$$

Using the isomorphism $H^{0,1}(X) \cong H^1(X,\mathcal{O}_X)$ we have that

$$J^1(X) \cong H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z})$$

where $H^1(X,\mathbb{Z})$ is naturally a subset of $H^1(X,\mathcal{O}_X)$ when viewed as cohomology groups of sheaves. Consider the the short exact sequence of sheaves defining the Chern class of line bundles

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{e^{(2\pi i\cdot)}} \mathcal{O}_X^* \longrightarrow 0$$

and the piece of the associated long exact sequence

$$H^1(X,\mathbb{Z}) \longrightarrow H^1(X,\mathcal{O}_X) \longrightarrow H^1(X,\mathcal{O}_X^*) \xrightarrow{c_1} H^2(X,\mathbb{Z}) .$$

The kernel of the Chern class map c_1 which we denote by $\mathrm{Pic}^0(X)$, is thus naturally isomorphic to

$$\mathrm{Pic}^0(X) \cong H^1(X,\mathcal{O}_X)/H^1(X,\mathbb{Z}) \cong J^1(X).$$

Classically, the first intermediate Jacobian of X is simply called the Jacobian of X and denoted by $J(X)$.

The Abel-Jacobi map in this case also has a geometric form. The domain of the map Φ_X^1 consists of cycles of codimension 1 or in other words divisors which are homologous to 0. A divisor D on X defines an isomorphism class of holomorphic line bundles $L_D \in H^1(X,\mathcal{O}_X^*)$. By theorem 11.33 in [5], we have that D is homologous to 0 if and only if $c_1(L_D) = 0$. This defines a map

$$\alpha : \mathcal{Z}^1(X)_{\text{hom}} \to \mathrm{Pic}^0(X) \cong J^1(X)$$

which sends a divisor D to L_D. We in fact have that $\alpha = \Phi_X^1$ (proposition 12.7 in [5]).

Example 2.6.3. Let X be a compact connected Riemann surface of genus g. It is possible to construct the Jacobian of X in a more concrete fashion. We have that

$$J(X) \cong \left(H^{1,0}(X)\right)^* / H_1(X,\mathbb{Z})$$

and the isomorphism $H^{1,0} \cong H^0(X, \Omega^1_X)$ gives us

$$J(X) \cong \frac{\{\text{holomorphic 1-forms on } X\}^*}{H_1(X, \mathbb{Z})}.$$

Fix $2g$ differentiable cycles $A_1, \ldots, A_g, B_1, \ldots B_g$ such that their classes in $H_1(X, \mathbb{Z})$ form a basis with respect to which the intersection matrix given by the cup product on $H^1(X, \mathbb{Z})$ and the isomorphism $H^1(X, \mathbb{Z}) \cong H_1(X, \mathbb{Z})$ is

$$\begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix}.$$

Such a basis for $H_1(X, \mathbb{Z})$ will be called standard. A basis $\{v_1, \ldots v_g\}$ of $H^0(X, \Omega^1_X)$ is called normalized if the matrix

$$\left(\int_{A_i} v_j \right)_{1 \le i,j \le g}$$

is the identity matrix and also the Riemann matrix defined by

$$\tau := \left(\int_{B_i} v_j \right)_{1 \le i,j \le g}$$

has the property that $\Im \tau$ is positive definite. Given a normalized basis $\{v_1, \ldots v_g\}$ of $H^0(X, \Omega^1_X)$ we have

$$J(X) = \mathbb{C}^g / \Lambda_\tau$$

where $\Lambda_\tau \subset \mathbb{C}^g$ is the lattice spanned by the columns of the matrix (I, τ). The Abel-Jacobi map takes the form

$$[p] - [q] \mapsto \left(\int_q^p v_i \right)_{1 \le i \le g} \in \mathbb{C}^g / \Lambda_\tau.$$

It can be seen here that the choice of the path of integration is irrelevant due to our choice of the lattice.

2.7 Singular Complex Curves and Jacobians.

It is possible to introduce complex curves abstractly as complex analytic spaces, but we will not need such generality.

Definition 2.7.1. Let W be a n-dimensional complex manifold. A complex curve is a compact analytic set $S \subset W$ of dimension one.

We have that a complex curve is smooth outside of finitely many points. We would like to generalize the notion of the Jacobian to complex curves. We have seen that the Jacobian of a smooth curve is naturally the kernel of the Chern class map.

Definition 2.7.2. For $S \subset W$ a complex curve, we define

$$J(S) = \ker \left(c_1 : H^1(S, \mathcal{O}_S^*) \to H^2(S, \mathbb{Z}) \right).$$

Let $S = X_1 \cup X_2 \subset W$ be a complex curve such that X_i are Riemann surfaces of genera g_i and such that $X_1 \cap X_2 = x$. We require also that there exists some set $U \subset W$ containing x, biholomorphic to D^2 where $D \subset \mathbb{C}$ is the open unit disc such that $S \cap U \cong \{(X, Y) \in D^2 | XY = 0\}$.

It is easy to verify that

$$H^1(S, \mathcal{O}_S^*) = H^1(X_1, \mathcal{O}_{X_1}^*) \oplus H^1(X_2, \mathcal{O}_{X_2}^*)$$

and

$$H^2(S, \mathbb{Z}) = H^2(X_1, \mathbb{Z}) \oplus H^2(X_2, \mathbb{Z}).$$

Moreover, the Chern class map preserves this decomposition. In particular we have

$$J(S) = J(X_1) \times J(X_2).$$

We can now define the Abel-Jacobi map Φ^1_S by the following property

$$\Phi^1_S([p] - [x]) = \begin{cases} \Phi^1_{X_1}([p] - [x]) \times \{0\} & \text{if } p \in X_1 \\ \{0\} \times \Phi^1_{X_2}([p] - [x]) & \text{if } p \in X_2 \end{cases}.$$

This defines the map Φ^1_S since the elements of the form $[p]-[x]$ generate the group $\mathcal{Z}^1(S)_{\mathrm{hom}}$. We can also describe the Abel-Jacobi map in terms of integrals analogously to how it was described in the previous section for smooth complex curves. Choose differentiable chains $A_1,\dots,A_{g_1},B_1,\dots B_{g_1}$ in X_1 and $A_{g_1+1},\dots,A_{g_1+g_2},B_{g_1+1},\dots,B_{g_1+g_2}$ in X_2 such that they form standard bases for $H_1(X_1,\mathbb{Z})$ and $H_1(X_2,\mathbb{Z})$ respectively. A collection $\{v_1,\dots,v_{g_1},v_{g_1+1},\dots v_{g_1+g_2}\}$ is called a normalized basis of holomorphic 1-forms on S if $\{v_1,\dots,v_{g_1}\}$ is a collection of holomorphic 1-forms on X_1 which forms a normalized basis of $H^0(X,\Omega^1_{X_1})$ with the Riemann matrix τ_1 and $\{v_{g_1+1},\dots,v_{g_1+g_2}\}$ is a collection of holomorphic 1-forms on X_2 which forms a normalized basis of $H^0(X,\Omega^1_{X_2})$ with the Riemann matrix τ_2. If we let $\tau=\begin{pmatrix}\tau_1 & 0\\ 0 & \tau_2\end{pmatrix}$, we have

$$J(S)=\mathbb{C}^{g_1+g_2}/\Lambda_\tau.$$

The Abel-Jacobi map then takes the form

$$[p]-[q]\mapsto\left(\int_q^p v_i\right)_{1\leq i\leq g_1+g_2}\in\mathbb{C}^{g_1+g_2}/\Lambda_\tau$$

where we define the value of an integral over a path in X_1 of a 1-form defined on X_2 to be zero (and vice versa).

3 Complex Tori and Line Bundles

In this section we will introduce some important results about complex tori which we will use in the following section in order to study intermediate Jacobians.

3.1 Complex Tori

By a complex torus we mean a complex Lie group X given by $X = V/\Lambda$ where V is a complex vector space and $\Lambda \subset V$ is a lattice of maximal rank in V. In fact, any connected compact complex Lie group is a complex torus. We can view V as the universal cover of X, and therefore we have a natural identification

$$\Lambda = \pi_1(X, 0).$$

Since $\pi_1(X, 0)$ is abelian, by the Hurewicz Theorem we have

$$H_1(X, \mathbb{Z}) \cong \Lambda.$$

Moreover, by the universal coefficient theorem, we have

$$H^1(X, \mathbb{Z}) \cong \mathrm{Hom}(H_1(X), \mathbb{Z}) \cong \mathrm{Hom}(\Lambda, \mathbb{Z}).$$

We have that X is homeomorphic to $(S^1)^{2n}$, and therefore using the Künneth formula, the cup product gives an isomorphism

$$H^n(X, \mathbb{Z}) \cong \bigwedge^n H^1(X, \mathbb{Z}) \cong \bigwedge^n \mathrm{Hom}(\Lambda, \mathbb{Z}) \cong \mathrm{Alt^n}(\Lambda, \mathbb{Z})$$

where $\mathrm{Alt}^n(\Lambda, \mathbb{Z})$ is the group of alternating n-forms on Λ with values in $\mathbb{Z}$.

By a homomorphism of complex tori we mean a homomorphism in the sense of complex Lie groups. Let $X = V/\Lambda$, $X' = V'/\Lambda'$ be two complex tori and let $f : X \to X'$ be a homomorphism. We have natural identifications $T_0X = V$, $T_0X' = V'$ of Lie algebras and universal coverings. The differential of f at 0 thus induces a map

$$F : V \to V'.$$

The exponential maps $\exp_{T_0X} : T_0X \to X$, $\exp_{T_0X'} : T_0X' \to X'$ in the sense of complex Lie groups are compatible with the maps F and f in the following way

$$f \circ \exp_{T_0X} = \exp_{T_0X'} \circ F.$$

Since in our case $\exp_{T_0X}$ coincides with the projection $V \to V/\Lambda$, we must have $F(\Lambda) \subset \Lambda'$. In particular we also have the map

$$F_\Lambda : \Lambda \to \Lambda'.$$

We call F the analytic representation of f and F_Λ the rational representation of f.

Definition 3.1.1. A homomorphism $f : X \to X'$ is an isogeny if it is surjective with a finite kernel. The exponent of f is defined as the exponent of the finite group $\ker f$.

It can be shown that isogeny defines an equivalence relation on complex tori.

3.2 Line Bundles and Factors of Automorphy

Let X be a compact complex manifold and let $\pi : \widetilde{X} \to X$ be the universal cover of X. If L is a holomorphic line bundle on X such that π^*L is trivial, there is a nice way to describe the isomorphism class of L using certain functions on $\widetilde{X}$.

Consider the fundamental group $\pi_1(X)$ of X as the group of automorphisms of coverings of $\widetilde{X}$. In particular we have a group action of $\pi_1(X)$ on $H^0(\mathcal{O}^*_{\widetilde{X}})$. The object of interest to us will be the first group cohomology $H^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}}))$. The group of cocycles $Z^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}}))$ is given by the functions $f : \pi_1(X) \times \widetilde{X} \to \mathbb{C}^*$ holomorphic in the second variable such that for all $\mu, \lambda \in \pi_1(X)$ and $\tilde{x} \in \widetilde{X}$,

$$f(\lambda\mu, \tilde{x}) = f(\lambda, \mu\tilde{x})f(\mu, \tilde{x}).$$

We also call those functions the factors of automorphy. The group of boundaries $B^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}}))$ is given by the functions

$$(\lambda, \tilde{x}) \mapsto h(\lambda\tilde{x})h(\tilde{x})^{-1}$$

where $h \in H^0(\mathcal{O}^*_{\widetilde{X}})$. The group cohomology $H^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}}))$ is defined by

$$H^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}})) = \frac{Z^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}}))}{B^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}}))}.$$

An element $f \in Z^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}}))$ defines a line bundle on X in the following way: we have the free and properly discontinuous action of $\pi_1(X)$ on $\widetilde{X} \times \mathbb{C}$ given by

$$\mu.(\tilde{x}, t) = (\mu\tilde{x}, f(\mu, \tilde{x})t)$$

which defines the line bundle $(\widetilde{X} \times \mathbb{C})/\pi_1(X) \to X$ on X. This correspondence gives us a homomorphism

$$\phi_1 : H^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}})) \to \mathrm{Pic}(X).$$

Clearly for any line bundle L in the image of ϕ_1, we have that π^*L is trivial. In fact ϕ_1 defines an isomorphism

$$\phi_1 : H^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}})) \to \ker\left(\pi^* : \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X})\right).$$

In other words, $H^1(\pi_1(X), H^0(\mathcal{O}^*_{\widetilde{X}}))$ is the group of line bundles on X which pull back to a trivial line bundle on $\widetilde{X}$.

With the use of factors of automorphy, one can associate global sections of a line bundle L on X with certain holomorphic functions on $\widetilde{X}$. Suppose L is a line bundle on X such that π^*L is trivial. Choose a trivialization of π^*L and take a factor of automorphy f corresponding to L and this trivialization (Picking a different trivialization corresponds to taking a different equivalent factor of automorphy). We have that global sections of L correspond to holomorphic functions ϑ on $\widetilde{X}$ satisfying

$$\vartheta(\lambda\tilde{x}) = f(\lambda, \tilde{x})\vartheta(\tilde{x})$$

for all $\lambda \in \pi_1(X)$ and $\tilde{x} \in \widetilde{X}$.

3.3 Line Bundles on Complex Tori

When $X = V/\Lambda$ is a complex torus, every line bundle on $\widetilde{X} = V$ is trivial since V is a complex vector space. We thus get that

$$H^1(\Lambda, H^0(V, \mathcal{O}^*_V)) = \mathrm{Pic}(X) = H^1(X, \mathcal{O}^*_X).$$

Moreover, there is a canonical way of assigning a factor of automorphy to a line bundle which we will now describe.

Definition 3.3.1. The Néron-Severi group of X is defined as the image of the Chern class map $c_1 : H^1(X, \mathcal{O}^*_X) \to H^2(X, \mathbb{Z})$.

$$NS(X) := c_1(H^1(X, \mathcal{O}^*_X)) \subset H^2(X, \mathbb{Z}).$$

Let L be a line bundle on X given by a factor of automorphy $f = \exp(2\pi i g)$, where $g\colon \pi_1(X) \times V \to \mathbb{C}$ is holomorphic in the second variable. The isomorphism $H^2(X, \mathbb{Z}) \to \mathrm{Alt}^2(\Lambda, \mathbb{Z})$ introduced in Section 3.1 has the property that it maps the first Chern class $c_1(L)$ of L to the alternating form

$$E_L(\lambda, \mu) = g(\mu, v + \lambda) + g(\lambda, v) - g(\lambda, v + \mu) - g(\mu, v)$$

where $\lambda, \mu \in \Lambda$ and $v \in V$ independently of the choice of g. We can extend E_L by $\mathbb{R}$-linearity to an alternating bilinear form on V.

Theorem 3.3.2. *Let $E : V \times V \to \mathbb{R}$ be an alternating bilinear form. We have that $E = E_L$ for some line bundle L on X if and only if there exists a hermitian form H on V satisfying $\Im H(\Lambda, \Lambda) \subset \mathbb{Z}$ and $E = \Im H$.*

This shows that $NS(X)$ could be seen as the group of hermitian forms H on V such that $\Im H(\Lambda) \subset \mathbb{Z}$.

Definition 3.3.3. Let $H \in NS(X)$ be a hermitian form. A semicharacter for H is a map $\chi : \Lambda \to U(1)$ where $U(1)$ is the multiplicative group of complex numbers of norm 1, satisfying

$$\chi(\lambda + \mu) = \chi(\lambda)\chi(\mu) \exp(\pi i \Im H(\lambda, \mu)).$$

Definition 3.3.4. Denote by $\mathcal{P}(\Lambda)$ the set of pairs (H, χ) where H is in $NS(X)$ and χ is a semicharacter for H.

For any $(H, \chi) \in \mathcal{P}(\Lambda)$, we can define a line bundle $L(H, \chi)$ by the following factor of automorphy,

$$a_{(H,\chi)}(\lambda, v) = \chi(\lambda) \exp(\pi H(v, \lambda) + \frac{\pi}{2} H(\lambda, \lambda)).$$

The Chern class of $L(H, \chi)$ is given by $\Im H$ restricted to Λ.

Theorem 3.3.5 (Appell-Humbert, [1, p. 32])**.** *The map $(H, \chi) \mapsto L(H, \chi)$ defines an isomorphism between $\mathcal{P}(\Lambda)$ and $\operatorname{Pic}(X)$.*

This gives us a canonical way of associating a factor of automorphy to an isomorphism class of line bundles. Namely the canonical factor of a line bundle $L = L(H, \chi)$ is $a_{(H,\chi)}$. Also of importance is that we can now express $\operatorname{Pic}^0(X)$ as the group of homomorphisms,

$$\operatorname{Pic}^0(X) \cong \operatorname{Hom}(\Lambda, U(1)).$$

We will need the following lemma in the later chapters.

Lemma 3.3.6. *Let $f : X \to X'$ be a homomorphism between two tori with analytic representation F and rational representation F_Λ. For all $(H, \chi) \in \mathcal{P}(\Lambda')$, we have*

$$f^* L(H, \chi) = L(F^* H, F_\Lambda^* \chi).$$

There is a canonical hermitian metric on the line bundle $L(H, \chi)$ for $(H, \chi) \in \mathcal{P}(\Lambda)$. We have that L is given by the quotient

$$(V \times \mathbb{C}) / \Lambda$$

where the action of Λ is defined using the canonical factor of automorphy $a_{(H,\chi)}$ by

$$\lambda.(v, t) = (v + \lambda, a_{(H,\chi)}(\lambda, v) \cdot t).$$

We will define a hermitian form on $L_{(H,\chi)}|_x$ for $x \in X$. Pick an element $v \in V$ which maps to x. The pullback of the projection map $(V \times \mathbb{C}) \to (V \times \mathbb{C}) / \Lambda$ defines an isomorphism

$$L_{(H,\chi)}|_x \cong \{v\} \times \mathbb{C} \cong \mathbb{C}.$$

Under this isomorphism, for two elements, $f, g \in \mathbb{C}$, we define

$$\langle f, g \rangle = f \overline{g} \exp(-\pi H(v, v)).$$

It is a matter of calculation to show that the hermitian form thus defined on $L_{(H,\chi)}|_x$ is independent of the choice of v.

3.4 Dual Complex Tori and The Poincaré Bundle

Let $X = V/\Lambda$ be a complex torus. Let $\hat{V} = \operatorname{Hom}_{\overline{\mathbb{C}}}(V, \mathbb{C})$ be the vector space of $\mathbb{C}$-antilinear forms on V. We have that $\hat{V}$ is naturally isomorphic to $\operatorname{Hom}_{\mathbb{R}}(V, \mathbb{R})$ as a real vector space with the isomorphism given by $l \mapsto \Im l$. We define the lattice, dual to Λ as

$$\hat{\Lambda} := \{l \in \hat{V} | \Im l(\Lambda) \subset \mathbb{Z}\}.$$

Definition 3.4.1. The torus dual to X is defined by

$$\hat{X} := \hat{V} / \hat{\Lambda}.$$

Definition 3.4.2. Given two tori $T_i = V_i/\Lambda_i$ and a homomorphism $f : T_1 \to T_2$ with the analytic representation $F : V_1 \to V_2$, the dual homomorphism

$$\hat{f} : \hat{T}_2 \to \hat{T}_1$$

is defined by its analytic representation $F^* : \hat{V}_2 \to \hat{V}_1$.

The homomorphism $\hat{V} \to \mathrm{Hom}(\Lambda, U(1))$ given by

$$l \mapsto \exp(2\pi i \Im l)$$

defines an isomorphism

$$\hat{X} \to \mathrm{Hom}(\Lambda,\mathrm{U}(1)) \cong \mathrm{Pic}^0(X).$$

We can thus expect there to be a line bundle $\mathcal{P}$ on $X \times \hat{X}$ that is in some sense universal.

Theorem 3.4.3. *There exists a line bundle $\mathcal{P}$ on $X \times \hat{X}$, unique up to isomorphism, such that*

$$\begin{array}{lll} \mathcal{P}|_{X\times\{L\}} & \simeq & L \qquad \forall L \in \hat{X} \cong Pic^0(X) \\ \mathcal{P}|_{\{0\}\times\hat{X}} & \textit{is} & \textit{trivial.} \end{array}$$

We call $\mathcal{P}$ the Poincaré bundle of X. The Appell-Humbert representation of the Poincaré bundle of X is the pair $(H, \chi) \in \mathcal{P}(\Lambda \oplus \hat{\Lambda})$ where H is the hermitian form on $V \oplus \hat{V}$ given by

$$H((v_1, l_1), (v_2, l_2) = \overline{l_2(v_1)} + l_1(v_2)$$

and $\chi : \Lambda \times \hat{\Lambda} \to U(1)$ is the semicharacter for H defined by

$$\chi(\lambda, l) = \exp\left(\pi i \Im l(\lambda)\right).$$

4 Mixed Intermediate co-Jacobians

In this section we will study the intermediate Jacobians of products of Kähler manifolds. The main tool for this section is the Künneth formula which describes the Hodge decomposition of the manifold $X \times Y$ in terms of Hodge decompositions of X and Y.

4.1 Mixed Intermediate co-Jacobians

We will introduce some new definitions in order to simplify notation. We recall that for X a compact connected Kähler manifold, the k-th intermediate Jacobian of X is

$$J^{2k-1}(X) \cong (F^{n-k+1}H^{2n-2k+1}(X))^*/H_{2n-2k+1}(X,\mathbb{Z})$$

where the dual space is the vector space of $\mathbb{C}$-linear forms. We have that the Abel-Jacobi map Φ_X^k sends analytic cycles of codimension k homologous to 0 to the k-th intermediate Jacobian of X.

Definition 4.1.1. We define the *k-th intermediate co-Jacobian* as

$$J_{2k-1}(X) = (F^kH^{2k-1}(X))^*/H_{2k-1}(X,\mathbb{Z}).$$

We have that

$$J_{2k-1}(X) \cong J^{2(n-k+1)-1}(X).$$

We define the co-Abel-Jacobi map as the appropriate Abel-Jacobi map,

$$\Phi_k^X := \Phi_X^{n-k+1}.$$

In particular we have that Φ_k^X maps analytic cycles of dimension $k-1$ homologous to 0 to the k-th intermediate co-Jacobian of X.

Definition 4.1.2. For X, Y compact Kähler manifolds and $k, l \in \mathbb{Z}_{\geq 0}$ such that $k+l$ is odd, we define the *mixed intermediate k, l co-Jacobian of the pair* (X, Y) as the torus

$$J_{k,l}(X,Y) = \left(\bigoplus_{i+j\geq \frac{k+l+1}{2}} H^{i,k-i}(X)\otimes H^{j,l-j}(Y)\right)^* / H_k(X,\mathbb{Z})\otimes H_l(Y,\mathbb{Z}).$$

Theorem 4.1.3. *Let X and Y be compact Kähler manifolds. For any positive integer k, we have the following decomposition of the k-th intermediate co-Jacobian of $X \times Y$*

$$J_{2k-1}(X\times Y) = \prod_{l+m=2k-1} J_{l,m}(X,Y).$$

Proof. The statement follows from the Künneth formula compatible with the Hodge decomposition. We have

$$H^{r,s}(X\times Y) = \bigoplus_{p+p'=r,q+q'=s} H^{p,q}(X)\otimes H^{p',q'}(Y)$$

given by the cup product. We thus have

$$\begin{aligned} F^kH^{2k-1}(X\times Y) &= \bigoplus_{r\geq k}\left(\bigoplus_{p+p'=r,q+q'=2k-1-r} H^{p,q}(X)\otimes H^{p',q'}(Y)\right) \\ &= \bigoplus_{l+m=2k-1}\left(\bigoplus_{i+j\geq\frac{l+m+1}{2}} H^{i,l-i}(X)\otimes H^{j,m-j}(Y)\right). \end{aligned}$$

For an abelian group A, we denote $A/\text{torsion}$ by A_0. We have the Künneth decomposition on the homology groups module torsion

$$H_{2k-1}(X\times Y,\mathbb{Z})_0 = \left(\bigoplus_{l+m=2k-1} H_l(X,\mathbb{Z})\otimes H_m(Y,\mathbb{Z})\right)_0$$

given for example by products of singular simplices. Since the action of the torsion elements in the homology groups with integral coefficients on the cohomology classes with complex coefficients is trivial we have

$$J_{2k-1}(X\times Y)=\bigoplus_{l+m=2k-1}\left(\bigoplus_{i+j\geq\frac{l+m+1}{2}}H^{i,l-i}(X)\otimes H^{j,m-j}(Y)\right)^{*}/\bigoplus_{l+m=2k-1}\left(H_l(X,\mathbb{Z})\otimes H_m(Y,\mathbb{Z})\right).$$

Now the action of $H_l(X,\mathbb{Z})\otimes H_m(Y,\mathbb{Z})$ on $H^{l'}(X)\otimes H^{m'}(Y)$ is non-trivial if and only if $l=l'$ and $m=m'$. In particular, when viewed as a subset of

$$(F^kH^{2k-1}(X\times Y))^{*}=\bigoplus_{l+m=2k-1}\left(\bigoplus_{i+j\geq\frac{l+m+1}{2}}H^{i,l-i}(X)\otimes H^{j,m-j}(Y)\right)^{*},$$

we have that $H_l(X,\mathbb{Z})\otimes H_m(Y,\mathbb{Z})\subset\left(\bigoplus_{i+j\geq\frac{l+m+1}{2}}H^{i,l-i}(X)\otimes H^{j,m-j}(Y)\right)^{*}$. In general, a torus $T=V/\Lambda$ splits as a product of n tori if there exist n vector subspaces $V_i\subset V$ such that $V=\bigoplus V_i$ and $\Lambda=\bigoplus\Lambda_i$ where $\Lambda_i:=\Lambda\cap V_i$. In this case we have that Λ_i is a full rank lattice in V_i and $T=\prod(V_i/\Lambda_i)$. In our case we have

$$\begin{aligned}J_{2k-1}(X\times Y)&=\bigoplus_{l+m=2k-1}\left(\bigoplus_{i+j\geq\frac{l+m+1}{2}}H^{i,l-i}(X)\otimes H^{j,m-j}(Y)\right)^{*}/\bigoplus_{l+m=2k-1}\left(H_l(X,\mathbb{Z})\otimes H_m(Y,\mathbb{Z})\right),\\&=\prod_{l+m=2k-1}\left(\bigoplus_{i+j\geq\frac{l+m+1}{2}}H^{i,l-i}(X)\otimes H^{j,m-j}(Y)\right)^{*}/\left(H_l(X,\mathbb{Z})\otimes H_m(Y,\mathbb{Z})\right),\\&=\prod_{l+m=2k-1}J_{l,m}(X,Y).\end{aligned}$$

□

It is natural to ask how the co-Abel-Jacobi map behaves with respect to this decomposition and how the intermediate co-Jacobians of X and Y relate to the mixed intermediate co-Jacobians of (X,Y). This is the content of the following theorems.

Theorem 4.1.4. *Let X,Y be compact Kähler manifolds. For Z_X a $(k-1)$-dimensional analytic cycle in X which is homologous to 0 and U_Y an arbitrary l-dimensional analytic cycle in Y, let $Z=Z_X\times U_Y$ be an analytic cycle in $X\times Y$. We have in particular that Z is homologous to 0 and*

$$\Phi_{k+l}^{X\times Y}(Z)\in J_{2k-1,2l}(X,Y).$$

Proof. Let Γ_X be a differentiable chain in X such that $\partial\Gamma_X=Z_X$. For $\Gamma=\Gamma_X\times U_Y$, we have that $Z=\partial\Gamma$ and the image of the co-Abel-Jacobi map is defined by

$$\Phi_{k+l}^{X\times Y}(Z)=\int_\Gamma .$$

Fix some Kähler metrics on X and Y. For any decomposable element

$$\alpha\otimes\beta\in(H^p(X,\mathbb{C})\otimes H^q(Y,\mathbb{C}))\cap(F^{k+l}H^{2k+2l-1}(X\times Y)),$$

we have

$$\int_\Gamma\alpha\otimes\beta=\int_{\Gamma_X\times U_Y}\tilde{\alpha}\wedge\tilde{\beta}$$

where $\tilde{\alpha}$ is the harmonic p-form on X with the class α and $\tilde{\beta}$ is the harmonic q-form on Y with the class β. In particular due to dimensions, we can see that $\int_\Gamma\alpha\otimes\beta=0$ if $p\neq 2k-1$ and $q\neq 2l$. In particular this implies that $\int_\Gamma\in\left(\bigoplus_{i+j\geq\frac{(2k-1)+(2l)+1}{2}}H^{i,(2k-1)-i}(X)\otimes H^{j,(2l)-j}(Y)\right)^{*}$ and thus $\Phi_{k+l}^{X\times Y}(Z)\in J_{2k-1,2l}(X,Y)$.

□

Clearly the above statement is true if we switch the roles of X and Y. In particular this shows that if U_Y is homologous to 0, then $\Phi_{k+l}^{X\times Y}(Z)=0$ since it must belong to $J_{2k-1,2l}(X,Y)$ and $J_{2k-2,2l+1}(X,Y)$.

Theorem 4.1.5. *Let U_Y be an analytic cycle in Y of dimension l and let k be a positive integer. There exists a homomorphism of tori*

$$\Psi_k^{U_Y}: J_{2k-1}(X)\to J_{2k-1,2l}(X,Y)$$

such that for all $Z_X\in \mathcal{Z}_{k-1}(X)_{\mathrm{hom}}$ we have

$$\Psi_k^{U_Y}\circ\Phi_k^X(Z_X)=\Phi_{k+l}^{X\times Y}(Z_X\times U_Y).$$

Moreover, $\Psi_k^{U_Y}$ depends only on the homology class of U_Y and has finite kernel if $[U_Y]$ is not a torsion element.

Proof. We have that

$$J_{2k-1}(X)=(F^kH^{2k-1}(X))^*/H_{2k-1}(X,\mathbb{Z})$$

and

$$J_{2k-1,2l}(X,Y)=\left(\bigoplus_{i+j\geq k+l}H^{i,2k-1-i}(X)\otimes H^{j,2l-j}(Y)\right)^*/H_{2k-1}(X,\mathbb{Z})\otimes H_{2l}(Y,\mathbb{Z}).$$

Consider the subspace S of the universal covering space of $J_{2k-1,2l}(X,Y)$ defined by

$$S:=\left(\bigoplus_{i\geq k}H^{i,2k-1-i}(X)\otimes H^{l,l}(Y)\right)^*=\left((F^kH^{2k-1}(X))\otimes H^{l,l}(Y)\right)^*.$$

We have that S is naturally a subspace of the universal covering space of $J_{2k-1,2l}(X,Y)$ since the dual of a direct sum of vector spaces is naturally the direct sum of the dual vector spaces. Consider now the following map

$$\begin{aligned}\overline{\Psi}_k^{U_Y}:(F^kH^{2k-1}(X))^* &\to S\\ \alpha &\mapsto \alpha\otimes\int_{U_Y}.\end{aligned}$$

Clearly $\overline{\Psi}_k^{U_Y}$ only depends on the homology class of U_Y. Now for $\alpha\in H_{2k-1}(X,\mathbb{Z})$, clearly $\alpha\otimes\int_{U_Y}\in H_{2k-1}(X,\mathbb{Z})\otimes H_{2l}(Y,\mathbb{Z})$. This shows that $\overline{\Psi}_k^{U_Y}$ induces a map of the tori $J_{2k-1}(X)\to J_{2k-1,2l}(X,Y)$ which we denote by $\Psi_k^{U_Y}$. Considering $\int_{U_Y}$ as an element of $\left(H^{l,l}(Y)\right)^*$ corresponds to taking the cup product with $[U_Y]$ which is an element of $H^{m-l,m-l}(Y,\mathbb{Z})$ where m is the dimension of Y. This shows that if $\int_{U_Y}=0$ as an element of $\left(H^{l,l}(Y)\right)^*$, then $[U_Y]$ is a torsion element. In particular, if $[U_Y]$ is not a torsion element, then $\overline{\Psi}_k^{U_Y}$ is injective and consequently $\Psi_k^{U_Y}$ has finite kernel.

We fix some Kähler metrics on X and Y. If C is some differentiable chain which is not necessarily closed and α is some cohomology class, by $\int_C\alpha$ we mean $\int_C\tilde{\alpha}$ where $\tilde{\alpha}$ is the harmonic form of class α. Let $Z_X\in\mathcal{Z}_{k-1}(X)_{\mathrm{hom}}$. We have that $Z_X=\partial\Gamma_X$ for some differentiable chain Γ_X in X and $\Phi_k^X(Z_X)=\int_{\Gamma_X}$. Consequently $\Psi_k^{U_Y}\circ\Phi_k^X(Z_X)$ is represented by $\int_{\Gamma_X}\otimes\int_{U_Y}$ when considered as an element in S. Note also that integration over U_Y is zero outside $H^{l,l}(Y)$ since U_Y is an analytic cycle of dimension l. Thus as an element of $\left(\bigoplus_{i+j\geq k+l}H^{i,2k-1-i}(X)\otimes H^{j,2l-j}(Y)\right)^*/H_{2k-1}(X,\mathbb{Z})\otimes H_{2l}(Y,\mathbb{Z})$, the element $\Psi_k^{U_Y}\circ\Phi_k^X(Z_X)$ is also represented by integrations:

$$\Psi_k^{U_Y}\circ\Phi_k^X(Z_X)=\int_{\Gamma_X}\otimes\int_{U_Y}.$$

If we let $\Gamma=\Gamma_X\times U_Y$ we have that $\partial\Gamma=Z_X\times U_Y$. In particular $\Phi_{k+l}^{X\times Y}(Z_X\times U_Y)$ is given by $\int_{\Gamma_X\times U_Y}$. By the explicit formulation of the Künneth decomposition, we have that for a decomposable element $\alpha\otimes\beta\in H^a(X,\mathbb{C})\otimes H^b(Y,\mathbb{C})\subset H^{a+b}(X\times Y,\mathbb{C})$ we have $\int_{\Gamma_X\times U_Y}\alpha\otimes\beta=\left(\int_{\Gamma_X}\alpha\right)\cdot\left(\int_{U_Y}\beta\right)$. This is the same as $\int_{\Gamma_X}\otimes\int_{U_Y}(\alpha\otimes\beta)$ and it shows that

$$\Phi_{k+l}^{X\times Y}(Z_X\times U_Y)=\int_{\Gamma_X}\otimes\int_{U_Y}=\Psi_k^{U_Y}\circ\Phi_k^X(Z_X).$$

□

Theorem 4.1.6. *If U_Y has dimension 0 and is not homologically trivial, then $\Psi_k^{U_Y}$ is an isogeny, and if U_Y is a point, then $\Psi_k^{U_Y}$ is an isomorphism.*

Proof. Note that if U_Y has dimension 0 and not homologically trivial, then $\Psi_k^{U_Y}$ is surjective since its kernel is finite and the target and the domain have the same dimension. What is left to show is that if U_Y is a point, then the kernel of $\Psi_k^{U_Y}$ is trivial. Let $U_Y = \{p\}$ and let $\alpha \in (F^k H^{2k-1}(X))^*$ be such that

$$\overline{\Psi}_k^{\{p\}}(\alpha) = \alpha \otimes \int_p \in H_{2k-1}(X) \otimes H_0(Y).$$

Here $\int_p$ can be simply viewed as the identity map on $H^{0,0}(Y) \cong \mathbb{C}$. For any cocycle $\beta \in F^k H^{2k-1}(X)$ such that $\beta + \overline{\beta} \in H^{2k-1}(X, \mathbb{Z})$, we have

$$\overline{\Psi}_k^{\{p\}}(\alpha)\,(\beta \otimes 1) \in \mathbb{Z}.$$

Since by definition we have

$$\overline{\Psi}_k^{\{p\}}(\alpha)\,(\beta \otimes 1) = \alpha\,(\beta)\,,$$

we conclude that $\alpha(\beta) \in \mathbb{Z}$ for all such β and thus $\alpha \in H_{2k-1}(X, \mathbb{Z})$. We therefore have that the map $\Psi_k^{\{p\}}$ is injective.

□

4.2 Duality of Mixed Intermediate co-Jacobians

We have that for an n-dimensional compact Kähler manifold X, the k-th intermediate co-Jacobian of X is dual to the $(n-k+1)$-th intermediate co-Jacobian of X

$$J_{2k-1}(X) \cong \hat{J}_{2(n-k+1)-1}(X)$$

where the duality is given by the pairing between $(F^k H^{2k-1}(X))^*$ and $(F^{n-k+1} H^{2(n-k+1)-1}(X))^*$, antilinear in the second term, which we will now define. We have the duality map

$$P : (H^{2(n-k+1)-1}(X, \mathbb{C}))^* \xrightarrow{\sim} H^{2k-1}(X, \mathbb{C})$$

given by the cup product and the complex conjugation map

$$^- : H^{2k-1}(X) \to H^{2k-1}(X).$$

For $\phi \in (F^k H^{2k-1}(X))^*$ and $\alpha \in (F^{n-k+1} H^{2(n-k+1)-1}(X))^*$, we define

$$[\phi, \alpha] = 2i \cdot \phi(\overline{P(\alpha)}).$$

Proposition 4.2.1. *Under this pairing, the lattice dual to $H_{2k-1}(X, \mathbb{Z}) \subset (F^k H^{2k-1}(X))^*$ is the lattice $H_{2(n-k+1)-1}(X, \mathbb{Z}) \subset (F^{n-k+1} H^{2(n-k+1)-1}(X))^*$.*

Proof. The proof of this fact is almost identical to the proof of the next theorem and thus will be omitted.

□

For X, Y compact Kähler manifolds, we have a decomposition of the intermediate co-Jacobians of $X \times Y$ as the products of mixed intermediate co-Jacobians of (X, Y). This decomposition preserves the duality introduced above.

Theorem 4.2.2. *Let X, Y be compact Kähler manifolds of dimensions n and m respectively. For any non-negative integers k, l such that $k + l$ is odd we have that $J_{k,l}(X, Y)$ is dual to $J_{2n-k,2m-l}(X, Y)$,*

$$J_{k,l}(X, Y) \cong \hat{J}_{2n-k,2m-l}(X, Y)$$

via the duality of $J_{k+l}(X \times Y)$ and $J_{2n+2m-k-l}(X \times Y)$.

Proof. We have

$$J_{k,l}(X,Y) = \left(\bigoplus_{i+j\geq \frac{k+l+1}{2}} H^{i,k-i}(X)\otimes H^{j,l-j}(Y)\right)^* / H_k(X,\mathbb{Z})\otimes H_l(Y,\mathbb{Z})$$

$$J_{2n-k,2m-l}(X,Y) = \left(\bigoplus_{i+j\geq \frac{k+l+1}{2}} H^{n-k+i,n-i}(X)\otimes H^{m-l+j,m-j}(Y)\right)^* / H_{2n-k}(X,\mathbb{Z})\otimes H_{2m-l}(Y,\mathbb{Z}),$$

where the indices in the expression of $J_{2n-k,2m-l}(X,Y)$ are rewritten in such a way that the summations in the two expressions are over the same set of values of i and j. The pairing we have between $(F^{\frac{k+l+1}{2}}H^{k+l}(X\times Y))^*$ and $(F^{\frac{2n+2m-k-l+1}{2}}H^{2n+2m-k-l}(X))^*$ restricts to a pairing between $\left(H^{n-k+i,n-i}(X)\otimes H^{m-l+j,m-j}(Y)\right)^*$ and $\left(H^{i,k-i}(X)\otimes H^{j,l-j}(Y)\right)^*$ for every i and j, antilinear in the second term. We have the duality map,

$$P : \left(H^{i,k-i}(X)\otimes H^{j,l-j}(Y)\right)^* \xrightarrow{\sim} \left(H^{n-i,n-k+i}(X)\otimes H^{m-j,m-l+j}(Y)\right)$$

given by the cup product and the complex conjugation map

$$\overline{} : H^{n-i,n-k+i}(X)\otimes H^{m-j,m-l+j}(Y) \xrightarrow{\sim} H^{n-k+i,n-i}(X)\otimes H^{m-l+j,m-j}(Y).$$

The pairing is given by

$$[\phi,\alpha] = 2i\cdot\phi(\overline{P(\alpha)})$$

for $\phi\in\left(H^{n-k+i,n-i}(X)\otimes H^{m-l+j,m-j}(Y)\right)^*$ and $\alpha\in\left(H^{i,k-i}(X)\otimes H^{j,l-j}(Y)\right)^*$. Since P is an isomorphism and $\overline{}$ is an anti-isomorphism, we see that $\left(H^{n-k+i,n-i}(X)\otimes H^{m-l+j,m-j}(Y)\right)^*$ is the entire space of antilinear forms of $\left(H^{i,k-i}(X)\otimes H^{j,l-j}(Y)\right)^*$. The same holds once we take the direct sum of the components of the universal covering space of $J_{k,l}(X,Y)$.

To show that $H_{2n-k}(X,\mathbb{Z})\otimes H_{2m-l}(Y,\mathbb{Z})$ is the lattice dual to $H_k(X,\mathbb{Z})\otimes H_l(Y,\mathbb{Z})$ it is convenient to write all of the spaces in question in terms of cohomologies, since there, it is clearer how the integral cohomologies are embedded in the subspaces of cohomologies with complex coefficients. Let

$$V = \bigoplus_{i+j\geq \frac{k+l+1}{2}} H^{n-i,n-k+i}(X)\otimes H^{m-j,m-l+j}(Y),$$

$$W = \bigoplus_{i+j\geq \frac{k+l+1}{2}} H^{k-i,i}(X)\otimes H^{l-j,j}(Y).$$

We have $V\subset H^{2n-k}(X,\mathbb{C})\otimes H^{2m-l}(Y,\mathbb{C})$ and $W\subset H^k(X,\mathbb{C})\otimes H^l(X,\mathbb{C})$. The conjugation on each cohomology group induces the conjugation on the tensor product and we have

$$H^{2n-k}(X,\mathbb{C})\otimes H^{2m-l}(Y,\mathbb{C}) = V\oplus\overline{V},$$

$$H^k(X,\mathbb{C})\otimes H^l(X,\mathbb{C}) = W\oplus\overline{W}.$$

The mixed intermediate co-Jacobians in question are then isomorphic to

$$J_{k,l}(X,Y) \cong V/H^{2n-k}(X,\mathbb{Z})_0\otimes H^{2m-l}(Y,\mathbb{Z})_0,$$

$$J_{2n-k,2m-l}(X,Y) \cong W/H^k(X,\mathbb{Z})_0\otimes H^l(Y,\mathbb{Z})_0.$$

Here, the integral cohomologies are considered as their projections onto the appropriate complex subspaces, i.e. the projection of $H^{2n-k}(X,\mathbb{Z})_0\otimes H^{2m-l}(Y,\mathbb{Z})_0\subset H^{2n-k}(X,\mathbb{R})\otimes H^{2m-l}(Y,\mathbb{R})$ onto V along $\overline{V}$ and the projection of $H^k(X,\mathbb{Z})_0\otimes H^l(Y,\mathbb{Z})_0$ onto W along $\overline{W}$. In this setting the pairing between V and W takes the form

$$[\phi,\alpha] = 2i\phi\wedge\alpha\in H^{2n}(X,\mathbb{C})\otimes H^{2l}(Y,\mathbb{C})\cong\mathbb{C}$$

where the last isomorphism is given by integration over X and Y.

Let ϕ be a projection of an element in $H^k(X,\mathbb{Z})_0\otimes H^l(Y,\mathbb{Z})_0$ onto W and let α be a projection of an element in $H^{2n-k}(X,\mathbb{Z})_0\otimes H^{2m-l}(Y,\mathbb{Z})_0$ onto V. Since integral cohomologies are real we have that $(\phi+\overline{\phi})\in H^k(X,\mathbb{Z})_0\otimes H^l(Y,\mathbb{Z})_0$ and $(\alpha+\overline{\alpha})\in H^{2n-k}(X,\mathbb{Z})_0\otimes H^{2m-l}(Y,\mathbb{Z})_0$. We thus have

$$(\phi+\overline{\phi})\wedge(\alpha+\overline{\alpha})\in\mathbb{Z}.$$

Due to type, we have that $\phi \wedge \alpha = \overline{\phi} \wedge \overline{\alpha} = 0$ and therefore $\phi \wedge \overline{\alpha} + \overline{\phi} \wedge \alpha = 2\Re(\phi \wedge \overline{\alpha}) \in \mathbb{Z}$. We thus have that $\Im([\phi, \alpha]) = \Im(2i\phi \wedge \overline{\alpha}) = 2\Re(\phi \wedge \overline{\alpha}) \in \mathbb{Z}$ and therefore ϕ lies in the dual lattice of the projection of $H^{2n-k}(X, \mathbb{Z})_0 \otimes H^{2m-l}(Y, \mathbb{Z})_0$ onto V.

To show the inverse inclusion, let $\phi \in W$ be such that for any projection α of an element in $H^{2n-k}(X, \mathbb{Z})_0 \otimes H^{2m-l}(Y, \mathbb{Z})_0$ onto V we have $\Im(2i\phi \wedge \overline{\alpha}) = 2\Re(\phi \wedge \overline{\alpha}) \in \mathbb{Z}$. We have that $(\phi + \overline{\phi})$ belongs to $H^k(X, \mathbb{R}) \otimes H^l(Y, \mathbb{R})$. An element $\gamma \in H^k(X, \mathbb{R}) \otimes H^l(Y, \mathbb{R})$ is integral if and only if for every $\delta \in H^{2n-k}(X, \mathbb{Z})_0 \otimes H^{2m-l}(Y, \mathbb{Z})_0$ we have $\gamma \wedge \delta \in \mathbb{Z}$. Let $\delta \in H^{2n-k}(X, \mathbb{Z})_0 \otimes H^{2m-l}(Y, \mathbb{Z})_0$ and let α be the projection of δ onto V. We have $\delta = \alpha + \overline{\alpha}$ and therefore

$$(\phi + \overline{\phi}) \wedge \delta = (\phi + \overline{\phi}) \wedge (\alpha + \overline{\alpha}).$$

Due to type, we have $\phi \wedge \alpha = \overline{\phi} \wedge \overline{\alpha} = 0$ and therefore

$$(\phi + \overline{\phi}) \wedge \delta = \overline{\phi} \wedge \alpha + \phi \wedge \overline{\alpha} = 2\Re(\phi \wedge \overline{\alpha}) \in \mathbb{Z}.$$

Since this is true for any $\delta \in H^{2n-k}(X, \mathbb{Z})_0 \otimes H^{2m-l}(Y, \mathbb{Z})_0$, we have $(\phi + \overline{\phi}) \in H^k(X, \mathbb{Z})_0 \otimes H^l(Y, \mathbb{Z})_0$ and ϕ is the projection of $(\phi + \overline{\phi})$ onto W.

□

4.3 The Poincaré Bundle on Products of Tori

We would like to know how the decomposition of the intermediate co-Jacobians into mixed intermediate co-Jacobians behaves with respect to the Poincaré bundles.

It is important to recall the explicit construction of the Poincaré bundle. Let $T = V/\Lambda$ be a torus and let $\hat{T} = \hat{V}/\hat{\Lambda}$ be its dual torus. We define a hermitian form H on $V \oplus \hat{V}$ by

$$H\left((v_1, l_1), (v_2, l_2)\right) = \overline{l_2(v_1)} + l_1(v_2),$$

and the semicharacter $\chi : \Lambda \oplus \hat{\Lambda} \to U(1)$ for H by

$$\chi(\lambda, \mu) = \exp(\pi i \Im \mu(\lambda)).$$

The pair (H, χ) defines the Poincaré bundle on $T \times \hat{T}$ by the canonical factor of automorphy $a_{\mathcal{P}} : (\Lambda \oplus \hat{\Lambda}) \times (V \oplus \hat{V}) \to \mathbb{C}^*$

$$a_{\mathcal{P}}\left((\lambda, \mu), (v, l)\right) = \chi(\lambda, \mu) \exp\left(\pi H\left((v, l), (\lambda, \mu)\right) + \frac{\pi}{2} H\left((\lambda, \mu), (\lambda, \mu)\right)\right).$$

We now investigate how the Poincaré bundle of a product of tori relates to the Poincaré bundles of individual tori.

Theorem 4.3.1. *Let T_1, T_2 be tori and let $\mathcal{P}$ be the Poincaré bundle on $(T_1 \times T_2) \times (\hat{T}_1 \times \hat{T}_2)$. For all $a \in T_2$ and $b \in \hat{T}_2$ we have that $\mathcal{P}|_{(T_1 \times \{a\}) \times (\hat{T}_1 \times \{b\})}$ is the Poincaré bundle on $(T_1 \times \{a\}) \times (\hat{T}_1 \times \{b\}) \cong T_1 \times \hat{T}_1$.*

Proof. Let $T_i = V_i/\Lambda_i$ and $\hat{T}_i = \hat{V}_i/\hat{\Lambda}_i$ for $i = 1, 2$. We write down explicitly the hermitian form H, the semicharacter χ for H and the canonical factor of automorphy $a_{\mathcal{P}}$ corresponding to $\mathcal{P}$. We have that H is the hermitian form on $V_1 \oplus V_2 \oplus \hat{V}_1 \oplus \hat{V}_2$ given by

$$H\left((v_1, v_2, l_1, l_2), (w_1, w_2, m_1, m_2)\right) = \overline{m_1(v_1)} + \overline{m_2(v_2)} + l_1(w_1) + l_2(w_2).$$

The semicharacter $\chi : \Lambda_1 \oplus \Lambda_2 \oplus \hat{\Lambda}_1 \oplus \hat{\Lambda}_2 \to U(1)$ for H is given by

$$\chi(\lambda_1, \lambda_2, \mu_1, \mu_2) = \exp(\pi i \Im\left(\mu_1(\lambda_1) + \mu_2(\lambda_2)\right)).$$

The canonical factor of automorphy of $\mathcal{P}$ is given by the map $a_{\mathcal{P}} : (\Lambda_1 \oplus \Lambda_2 \oplus \hat{\Lambda}_1 \oplus \hat{\Lambda}_2) \times (V_1 \oplus V_2 \oplus \hat{V}_1 \oplus \hat{V}_2) \to \mathbb{C}^*$

$$\begin{aligned} a_{\mathcal{P}}\left((\lambda_1, \lambda_2, \mu_1, \mu_2), (v_1, v_2, l_1, l_2)\right) &= \chi(\lambda_1, \lambda_2, \mu_1, \mu_2) \exp\left(\pi H\left((v_1, v_2, l_1, l_2), (\lambda_1, \lambda_2, \mu_1, \mu_2)\right)\right) \cdot \\ &\quad \exp\left(\frac{\pi}{2} H\left((\lambda_1, \lambda_2, \mu_1, \mu_2), (\lambda_1, \lambda_2, \mu_1, \mu_2)\right)\right). \end{aligned}$$

Let $v_2 \in V_2$ map to a and $l_2 \in \hat{V}_2$ map to b. We have that $(T_1 \times \{a\}) \times (\hat{T}_1 \times \{b\})$ is the torus given by $(V_1 \times \{v_2\}) \oplus (\hat{V}_1 \times \{l_2\}) / \left(\Lambda_1 \oplus \hat{\Lambda}_1\right)$. Here, we consider the set $(V_1 \times \{v_2\})$ as the vector space naturally isomorphic to V_1 and analogously $(\hat{V}_1 \times \{l_2\})$ as the vector space naturally isomorphic to $\hat{V}_1$. As such we have that $(V_1 \times \{v_2\}) \oplus (\hat{V}_1 \times \{l_2\})$ is a subset but not in general a subspace of $(V_1 \oplus V_2) \oplus (\hat{V}_1 \oplus \hat{V}_2)$. We have that the factor of automorphy of the restriction of $\mathcal{P}$ to $(T_1 \times \{a\}) \times (\hat{T}_1 \times \{b\})$ is given by the restriction of $a_{\mathcal{P}}$ to $\left(\Lambda_1 \oplus \{0\} \oplus \hat{\Lambda}_1 \oplus \{0\}\right) \times \left((V_1 \times \{v_2\}) \oplus (\hat{V}_1 \times \{l_2\})\right)$

$$\begin{aligned} a_{\mathcal{P}}\left((\lambda_1, 0, \mu_1, 0), (v_1, v_2, l_1, l_2)\right) &= \chi(\lambda_1, 0, \mu_1, 0) \exp\left(\pi H\left((v_1, v_2, l_1, l_2), (\lambda_1, 0, \mu_1, 0)\right)\right) \cdot \\ &\quad \exp\left(\frac{\pi}{2} H\left((\lambda_1, 0, \mu_1, 0), (\lambda_1, 0, \mu_1, 0)\right)\right) \\ &= a_{\mathcal{P}'}\left((\lambda_1, \mu_1), (v_1, l_1)\right) \end{aligned}$$

where $\mathcal{P}'$ is the Poincaré bundle on $T_1 \times \hat{T}_1$.

□

Let X, Y be compact Kähler manifolds of dimensions n and m respectively. Let $k, a, b \in \mathbb{Z}$ such that $a + b = 2k - 1$. By Theorems 4.3.1 and 4.2.2 we have that the Poincaré bundle on $J_{2k-1}(X \times Y) \times J_{2(n+m-k+1)-1}(X \times Y)$ restricted to $J_{a,b}(X, Y) \times J_{2n-a,2m-b}(X, Y)$ is the Poincaré bundle on $J_{a,b}(X, Y) \times J_{2n-a,2m-b}(X, Y)$. Let U_1, U_2 be analytic cycles in Y of dimensions l and $m - l$ respectively. We have the following diagram

$$\begin{tikzcd} J_{2k-1}(X) \arrow[r, "\Psi_k^{U_1}"] \arrow[d, leftrightsquigarrow, "\text{duality}"] & J_{2k-1,2l}(X, Y) \arrow[d, leftrightsquigarrow, "\text{duality}"] \\ J_{2n-2k+1}(X) \arrow[r, "\Psi_{n-k+1}^{U_2}"] & J_{2n-2k+1,2m-2l}(X, Y) \end{tikzcd}$$

and thus we have the Poincaré bundles on the left and right columns. We would want to know if they are compatible under the horizontal maps.

Lemma 4.3.2. *Let T_1, T_2 be tori and let $\phi : T_1 \to T_2$, $\psi : \hat{T}_1 \to \hat{T}_2$ be homomorphisms of tori. Let $\mathcal{P}_1, \mathcal{P}_2$ be the Poincaré bundles on $T_1 \times \hat{T}_1$ and $T_2 \times \hat{T}_2$ respectively. We then have that $(\phi, \psi)^* \mathcal{P}_2 \cong \mathcal{P}_1$ if and only if $\hat{\phi} \circ \psi = id$.*

Proof. Let $f : X \to Y$ be a homomorphism of tori with the analytic representation F. Denote by $L(H, \chi)$ the line bundle corresponding to a hermitian form H and a semicharacter χ for H. By Lemma 3.3.6, we have

$$f^* L(H, \chi) = L(F^* H, F^* \chi).$$

Let Φ and Ψ be the analytic representations of ϕ and ψ respectively and let $\mathcal{P}_i = L(H_i, \chi_i)$. We have that $(\phi, \psi)^* \mathcal{P}_2 \cong \mathcal{P}_1$ if and only if

$$\begin{aligned} (\Phi, \Psi)^* H_2 &= H_1, \\ (\Phi, \Psi)^* \chi_2 &= \chi_1. \end{aligned}$$

We have that $(\Phi, \Psi)^* H_2 = H_1$ if and only if for all $v_1, v_2 \in V_1$ and $l_1, l_2 \in \hat{V}_1$ we have

$$\overline{\Psi(l_2)\left(\Phi(v_1)\right)} + \Psi(l_1)\left(\Phi(v_2)\right) = \overline{l_2(v_1)} + l_1(v_2)$$

which is equivalent to the statement that for all $v \in V_1$ and $l \in \hat{V}_1$,

$$\Psi(l_1)\left(\Phi(v_2)\right) = l_1(v_2)$$

which in turn is equivalent to the fact that $\Phi^* \Psi = \text{id}$. If $\Phi^* \Psi = \text{id}$, then clearly

$$(\Phi, \Psi)^* \chi_2(\lambda, \mu) = \exp(\pi i \Im(\Phi^* \Psi \mu(\lambda)) = \chi_1(\lambda, \mu).$$

We have that Ψ^* is the analytic representation of $\hat{\psi}$, therefore $\Psi^* \Phi = \text{id}$ if and only if $\hat{\phi} \circ \psi = \text{id}$.

□

Theorem 4.3.3. *Let X, Y be compact Kähler manifolds of dimensions n and m respectively. Let U_1, U_2 be analytic cycles in Y of dimensions l and $m-l$ respectively. We have that the pullback of the map*

$$J_{2k-1}(X) \times J_{2n-2k+1}(X) \xrightarrow{\Psi_k^{U_1} \times \Psi_{n-k+1}^{U_2}} J_{2k-1,2l}(X,Y) \times J_{2n-2k+1,2m-2l}(X,Y)$$

sends the Poincaré bundle on the image to the Poincaré bundle on the domain if and only if $[U_1] \wedge [U_2] = 1$.

Proof. It follows from Lemma 4.3.2 that the pullback of the map $\Psi_k^{U_1} \times \Psi_{n-k+1}^{U_2}$ sends the Poincaré bundle to the Poincaré bundle if and only if $(\overline{\Psi}_{n-k+1}^{U_2})^* \circ \overline{\Psi}_k^{U_1} = \mathrm{id}$ where $\overline{\Psi}_{n-k+1}^{U_2}, \overline{\Psi}_k^{U_1}$ are the analytic representations of $\Psi_{n-k+1}^{U_2}$ and $\Psi_k^{U_1}$ respectively. This is equivalent to the fact that for all

$$\alpha \in (F^k H^{2k-1}(X))^* \cong \bigoplus_{i \geq n-k+1} H^{2n-2k+1-i,i}(X)$$

and

$$\beta \in (F^{n-k+1} H^{2n-2k+1}(X))^*,$$

we have

$$[(\beta \otimes [U_1]), (\alpha \otimes [U_2])] = [\beta, \alpha] = 2i\beta(\overline{\alpha}).$$

We have by the definition of the pairing between

$$\left(\bigoplus_{i+j \geq \frac{2k+l}{2}} H^{n-2k+1+i,n-i}(X) \otimes H^{m-l+j,m-j}(Y) \right)^*$$

and

$$\left(\bigoplus_{i+j \geq \frac{2k+l}{2}} H^{i,2k-1-i}(X) \otimes H^{j,l-j}(Y) \right)^*$$

that

$$[(\beta \otimes [U_1]), (\alpha \otimes [U_2])] = 2i\beta(\overline{\alpha}) \cdot [U_1] \wedge \overline{[U_2]}.$$

Since $[U_2]$ is a real class, the statement of the theorem follows.

□

Corollary 4.3.4. *Let X, Y be compact Kähler manifolds of dimensions n and m respectively. Let U_1 be an analytic cycle in Y of dimension l such that there exists an analytic cycle U_2 of Y of complementary dimension such that $[U_1] \wedge [U_2] = 1$. We then have that $\Psi_k^{U_1}$ is injective.*

Proof. By Theorem 4.3.3 and Lemma 4.3.2 we have that $(\hat{\Psi_{n-k+1}^{U_2}}) \circ \Psi_k^{U_1} = \mathrm{id}$ and therefore $\Psi_k^{U_1}$ is injective.

□

4.4 Intermediate co-Jacobians of Hodge Structures

Intermediate co-Jacobians can be defined more generally for any integral Hodge structure. We can therefore repeat most of the previous constructions for arbitrary Hodge structures, their products and their duals.

Definition 4.4.1. Let $V = (V_{\mathbb{Z}}, V^{p,q})$ be an integral Hodge structure of weight $2k-1$. We define the intermediate co-Jacobian of V in the following way

$$J_{2k-1}(V) = (F^k V_{\mathbb{C}})^* / (V_{\mathbb{Z}})^*$$

where $F^k V_{\mathbb{C}} = \bigoplus_{p \geq k} V^{p,q}$ and $(V_{\mathbb{Z}})^* = \mathrm{Hom}(V_{\mathbb{Z}}, \mathbb{Z})$ which by $\mathbb{C}$-linearity extends to a subset of $V_{\mathbb{C}}^*$.

Theorem 4.4.2. *We have that $J_{2k-1}(V)$ is a complex torus.*

Proof. Observe that $V_{\mathbb{C}} = (F^k V_{\mathbb{C}}) \oplus \overline{(F^k V_{\mathbb{C}})}$. In particular this shows that $\dim_{\mathbb{R}}(F^k V_{\mathbb{C}}) = \operatorname{rank}(V_{\mathbb{Z}})$. Thus the image of $(V_{\mathbb{Z}})^*$ under the map

$$i : (V_{\mathbb{Z}})^* \to (F^k V_{\mathbb{C}})^*$$

has full rank if and only if i is injective. Let $\alpha \in (V_{\mathbb{Z}})^*$ be in the kernel of i. It follows that for all $v \in (F^k V_{\mathbb{C}})$, we have $\alpha(v) = 0$. Since for any $w \in V_{\mathbb{C}}$, we have $\alpha(\overline{w}) = \overline{\alpha(w)}$, and $V_{\mathbb{C}} = (F^k V_{\mathbb{C}}) \oplus \overline{(F^k V_{\mathbb{C}})}$, we conclude that α is trivial on the entire $V_{\mathbb{C}}$. In particular α is trivial on $V_{\mathbb{Z}} \subset V_{\mathbb{C}}$ and therefore $\alpha = 0$. We thus conclude that $(V_{\mathbb{Z}})^*$ is a lattice of maximal rank in $(F^k V_{\mathbb{C}})^*$ and $J_{2k-1}(V)$ is a complex torus. □

Definition 4.4.3. Let $V = (V_{\mathbb{Z}}, V^{p,q})$ and $W = (W_{\mathbb{Z}}, W^{p,q})$ be integral Hodge structures of weights k, l such that $k + l$ is odd. We define the mixed intermediate co-Jacobian of V and W as

$$J_{k,l}(V, W) = \left(\bigoplus_{i+j \geq \frac{k+l+1}{2}} V^{i,k-i} \otimes W^{j,l-j} \right)^* / \, (V_{\mathbb{Z}} \otimes W_{\mathbb{Z}})^* .$$

Theorem 4.4.4. *Let $V = (V_{\mathbb{Z}}, V^{p,q})$ and $W = (W_{\mathbb{Z}}, W^{p,q})$ be integral Hodge structures of weights k, l such that $k + l$ is odd. We have*

$$J_{k,l}(V, W) = J_{k+l}(V \otimes W).$$

Proof. This follows immediately from the definition of the tensor product of Hodge structures

$$\begin{aligned} (V \otimes W)_{\mathbb{Z}} &= V_{\mathbb{Z}} \otimes W_{\mathbb{Z}}, \\ (V \otimes W)^{r,s} &= \bigoplus_{p+p'=r, q+q'=s} V^{p,q} \otimes W^{p',q'}. \end{aligned}$$

□

Theorem 4.4.5. *Let $V = (V_{\mathbb{Z}}, V^{p,q})$ and $W = (W_{\mathbb{Z}}, W^{p,q})$ be integral Hodge structures of weight $2k - 1$. We have*

$$J_{2k-1}(V \oplus W) = J_{2k-1}(V) \times J_{2k-1}(W).$$

Proof. We have that

$$J_{2k-1}(V \oplus W) = (F^k V_{\mathbb{C}})^* \oplus (F^k W_{\mathbb{C}})^* / (V_{\mathbb{Z}})^* \oplus (W_{\mathbb{Z}})^*.$$

Since the action of $(V_{\mathbb{Z}})^*$ is 0 on $W_{\mathbb{C}}$, the abelian group $(V_{\mathbb{Z}})^* \oplus \{0\}$ as a subset of $(F^k V_{\mathbb{C}})^* \oplus (F^k W_{\mathbb{C}})^*$, lies in the subspace $(F^k V_{\mathbb{C}})^* \oplus \{0\}$. By the same argument we have $\{0\} \oplus (W_{\mathbb{Z}})^* \subset \{0\} \oplus (F^k W_{\mathbb{C}})^*$ and therefore $J_{2k-1}(V \oplus W)$ splits as a product of two tori,

$$\begin{aligned} J_{2k-1}(V \oplus W) &= \big((F^k V_{\mathbb{C}})^* / (V_{\mathbb{Z}})^*\big) \times \big((F^k W_{\mathbb{C}})^* / (W_{\mathbb{Z}})^*\big) \\ &= J_{2k-1}(V) \times J_{2k-1}(W). \end{aligned}$$

□

Definition 4.4.6. Let $V = (V_{\mathbb{Z}}, V^{p,q})$ be an integral Hodge structure of weight k. We define the dual Hodge structure V^* of weight k as

$$\begin{aligned} (V^*)_{\mathbb{Z}} &= (V_{\mathbb{Z}})^*, \\ (V^*)^{p,q} &= (V^{q,p})^*. \end{aligned}$$

We naturally have that $V_{\mathbb{Z}}^* \otimes \mathbb{C} \cong (V_{\mathbb{Z}} \otimes \mathbb{C})^*$, therefore this construction indeed defines a Hodge structure. The conjugation on $(V_{\mathbb{C}})^*$ coming from this isomorphism has the following form: for $\phi \in (V_{\mathbb{C}})^*$ and $x \in V_{\mathbb{C}}$, we have

$$\overline{\phi}(x) = \overline{\phi(\overline{x})}.$$

Theorem 4.4.7. *For $V = (V_{\mathbb{Z}}, V^{p,q})$ an integral Hodge structure of weight $2k - 1$, there is a natural isomorphism*

$$J_{2k-1}(V^*) \cong \hat{J}_{2k-1}(V).$$

Proof. We have that the universal covering space of $J_{2k-1}(V^*)$ is $(F^k V^*_{\mathbb{C}})^* = \bigoplus_{p\geq k}(V^{q,p})$. Define a pairing between $(F^k V^*_{\mathbb{C}})^* = \bigoplus_{p\geq k}(V^{q,p})$ and $(F^k V_{\mathbb{C}})^* = \bigoplus_{p\geq k}(V^{p,q})^*$, antilinear in the second term, by

$$[\alpha, x] = 2i \cdot \overline{x(\overline{\alpha})}$$

where $\alpha \in (F^k V^*_{\mathbb{C}})^*$ and $x \in (F^k V_{\mathbb{C}})^*$. Clearly this pairing is perfect and we only have to check that the lattice, dual under this pairing to the image of $V^*_{\mathbb{Z}}$ in $(F^k V_{\mathbb{C}})^*$, is exactly the image of $V_{\mathbb{Z}}$ in $(F^k V^*_{\mathbb{C}})^*$.

Let $\alpha \in (F^k V^*_{\mathbb{C}})^*$ and $x \in (F^k V_{\mathbb{C}})^*$ lie in the images of $V_{\mathbb{Z}}$ and $V^*_{\mathbb{Z}}$ respectively. We have that $(\alpha + \overline{\alpha}) \in V_{\mathbb{Z}}$, $(x + \overline{x}) \in V^*_Z$ and therefore

$$[(\alpha + \overline{\alpha}), (x + \overline{x})] \in 2i\mathbb{Z}.$$

Since $\alpha \in \bigoplus_{p\geq k}(V^{q,p}) = \overline{F^k V_{\mathbb{C}}}$ and $x \in (F^k V_{\mathbb{C}})^*$, we have by definition of the pairing,

$$[\overline{\alpha}, x] = 2i \cdot \overline{x(\alpha)} = 0.$$

Similarly we deduce that $[\alpha, \overline{x}] = 0$. We thus have that

$$[(\alpha + \overline{\alpha}), (x + \overline{x})] = [\alpha, x] + [\overline{\alpha}, \overline{x}] = 2i \cdot 2\Re\left(\overline{x(\overline{\alpha})}\right).$$

It follows that $2\Re(\overline{x(\overline{\alpha})}) \in \mathbb{Z}$, which implies that $\Im(2i \cdot \overline{x(\overline{\alpha})}) = \Im([\alpha, x]) \in \mathbb{Z}$. This means that α lies in the lattice, dual to the image of $V^*_{\mathbb{Z}}$ in $(F^k V_{\mathbb{C}})^*$.

For the other inclusion, assume that $\alpha \in (F^k V^*_{\mathbb{C}})^*$ lies in the lattice, dual to the image of $V^*_{\mathbb{Z}}$ in $(F^k V_{\mathbb{C}})^*$. We will show that $(\alpha + \overline{\alpha}) \in V_{\mathbb{Z}}$. By the assumption, for all $x \in (F^k V_{\mathbb{C}})^*$ such that $(x + \overline{x}) \in V^*_{\mathbb{Z}}$, we have

$$\Im([\alpha, x]) \in \mathbb{Z}.$$

We have to show that $(x+\overline{x})\,(\alpha + \overline{\alpha}) \in \mathbb{Z}$. By the same argument as above, we have that $x(\alpha) = \overline{x}(\overline{\alpha}) = 0$. We thus have to show that $2\Re(x(\overline{\alpha})) \in \mathbb{Z}$. Since we have $2\Re(x(\overline{\alpha})) = \Im(2i \cdot \overline{x(\overline{\alpha})}) = \Im([\alpha, x])$, the statement is proved.

□

5 Ceresa Cycles and Degenerate Complex Curves

In this section we give an example of when we are interested in intermediate Jacobians of a product of two Kähler manifolds.

5.1 Ceresa Cycles

Given a Riemann surface S and a point $q \in S$, we can construct a map $a_q : S \to J(S)$ given by

$$p \mapsto \Phi_1^S([p] - [q]) = \int_q^p .$$

We define the Ceresa cycle of S as

$$C_q(S) = a_q(S) - [-1]^* a_q(S).$$

It is an analytic cycle in $J(X)$. For a torus $T = V/\Lambda$ we have $H_k(T) \cong \bigwedge^k \Lambda$ and we can thus see that $[-1]^*$ acts on $H_k(T)$ as $(-1)^k$. In particular $[-1]^*$ is the identity map on $H_2(T)$ and thus $\langle a_q(S)\rangle = \langle [-1]^* a_q(S)\rangle$. In our case this shows that the Ceresa cycle $C_q(S)$ is homologically trivial. Let $X := J(S)$ and consider the image of the Ceresa cycle under the map Φ_2^X. We define the Ceresa element of S as

$$c_q(S) := \Phi_2^X(C_q(S)) \in J_3(X).$$

Consider now a degenerate case of a complex curve as in section 2.7 where the curve S is the union of two Riemann surfaces M and N intersecting transversely at a point q. Let $X = J(N)$ and $Y = J(M)$. We have shown that in a natural way

$$J(S) = X \times Y$$

and have defined the Abel-Jacobi map for S. We define the Ceresa cycle and the Ceresa element for S with respect to the point q in the same manner as before. In this case, the Ceresa element of S lies in

$$J_3(X \times Y) = J_{3,0}(X, Y) \times J_{2,1}(X, Y) \times J_{1,2}(X, Y) \times J_{0,3}(X, Y).$$

Theorem 5.1.1. *We have $c_q(S) = \Psi_2^{\{0\}}(c_q(N)) + \Psi_2^{\{0\}}(c_q(M))$. The first summand belongs to $J_{3,0}(X, Y)$ and the second belongs to $J_{0,3}(X, Y)$.*

Proof. We have that

$$\begin{aligned} C_q(S) &= a_q(S) - [-1]^* a_q(S) \\ &= (a_q(N) \times \{0\}) \cup (\{0\} \times a_q(M)) - [-1]^* \left(a_q(N) \times \{0\}) \cup (\{0\} \times a_q(M)\right). \end{aligned}$$

Since $(a_q(N) \times \{0\}) \cup (\{0\} \times a_q(M)) = (a_q(N) \times \{0\}) + (\{0\} \times a_q(M))$ as a cycle we conclude that

$$C_q(S) = C_q(N) \times \{0\} + \{0\} \times C_q(M).$$

Therefore by Theorem 4.1.5

$$\begin{aligned} \Phi_2^{X\times Y}(C_q(N) \times \{0\} + \{0\} \times C_q(M)) &= \Phi_2^{X\times Y}(C_q(N) \times \{0\}) + \Phi_2^{X\times Y}(\{0\} \times C_q(M)) \\ &= \Psi_2^{\{0\}}(c_q(N)) + \Psi_2^{\{0\}}(c_q(M)). \end{aligned}$$

□

5.2 A Degenerate Family of Complex Curves

A singular complex curve can occur naturally as a fiber in an analytic family whose generic fiber is a smooth Riemann surface. In this section we will present a complex analytic family of complex curves

$$\pi_{\mathcal{X}} : \mathcal{X} \to D$$

where D is the complex unit disc with a degenerate fiber over the point $0 \in D$ as presented in [2]. Restricting to some neighborhood D_ϵ of 0 in D, we will construct the family of Jacobians corresponding to this family which will have all non-singular fibers

$$\pi_{\mathcal{J}} : \mathcal{J} \to D_\epsilon.$$

We also have the family

$$\pi_{\mathcal{J}_3(\mathcal{J})} : \mathcal{J}_3(\mathcal{J}) \to D_\epsilon$$

whose fiber over a point $t \in D_\epsilon$ is the second intermediate co-Jacobian of the fiber of $\pi_{\mathcal{J}}$ over t. Given a holomorphic section

$$q : D_\epsilon \to \mathcal{X}$$

we can construct the section

$$c_q(\mathcal{X}) : D_\epsilon \to \mathcal{J}_3(\mathcal{J})$$

which maps a point $t \in D_\epsilon$ to the Ceresa element of $\pi_{\mathcal{X}}^{-1}(t)$ with respect to $q(t)$. We will show that the section $c_q(\mathcal{X})$ is holomorphic and thus the Ceresa element varies holomorphically everywhere.

Let C_1 and C_2 be two Riemann surfaces of genera g_1 and g_2 respectively. Let $p_1 \in C_1$ and $p_2 \in C_2$ be two fixed points and let U_1, U_2 be open neighborhoods of p_1 and p_2 biholomorphic to the open unit disc $D \subset \mathbb{C}$. Let z_i be a holomorphic coordinate on U_i centered at p_i for $i = 1, 2$. Consider the sets

$$W_i = (C_i \times D) \setminus \{(x, t) \in U_i \times D \text{ s.t. } |z_i(x)| \leq |t|\}.$$

The sets W_i are complex manifolds being open subsets of $C_i \times D$. We naturally have projection maps $\pi_i : W_i \to D$ whose fibers over $t \in D$ are the Riemann surfaces C_i with a puncture around the points p_i. We will "glue" the sets W_1 and W_2 in a way to produce a family $\mathcal{X} \to D$ whose fiber over $t \neq 0$ is the gluing of $\pi_i^{-1}(t)$ along the neighborhoods of the punctures and over $t = 0$ the fiber is the union of C_1 and C_2 intersecting transversely at the points p_1 and p_2. The general fiber will be a Riemann surface of genus $g_1 + g_2$.

Let $S \subset D^3$ be the surface given by

$$XY = t$$

where X, Y, t are the coordinates on D^3. Define

$$\mathcal{X} := W_1 \sqcup S \sqcup W_2 / \sim$$

where the relation $\sim$ is given by

$$(x, t) \sim (z_1(x), \frac{t}{z_1(x)}, t)$$

for $(x, t) \in (U_1 \times D) \cap W_1$ and

$$(x, t) \sim (\frac{t}{z_2(x)}, z_2(x), t)$$

for $(x, t) \in (U_2 \times D) \cap W_2$. These relations are consistent with the projections onto D and we therefore get a family

$$\pi_{\mathcal{X}} : \mathcal{X} \to D.$$

The map $\pi_{\mathcal{X}}$ is flat, proper and a submersion outside of $\pi_{\mathcal{X}}^{-1}(0)$. Denote $\pi_{\mathcal{X}}^{-1}(t)$ by C_t. We call $S \subset \mathcal{X}$ the pinching region of the family. For $t \neq 0$ we have that $C_t \cap S$ is the region where $C_t \cap (U_i \times D)$ are glued holomorphically. For $t = 0$, we have $C_0 \cap S \cong \{(X, Y) \in D^2 | XY = 0\}$ where the sheet $Y = 0$ without the origin is glued to $U_1 \backslash \{p_1\} \times \{0\} \subset W_1$ and the sheet $X = 0$ without the origin is glued to $U_2 \backslash \{p_2\} \times \{0\} \subset W_2$.

In order to construct the family of Jacobians of $\mathcal{X}$, we will use the explicit construction of the Jacobian as introduced in section 2.7. Choose differentiable chains $A_1, \ldots, A_{g_1}, B_1, \ldots B_{g_1}$ in C_1 and $A_{g_1+1}, \ldots, A_{g_1+g_2}, B_{g_1+1}, \ldots, B_{g_1+g_2}$ in C_2 such that they form standard bases for $H_1(C_1, \mathbb{Z})$ and $H_1(C_2, \mathbb{Z})$ respectively as described in Section 2.6 and such that they don't intersect U_1 and U_2. For any $t \in D$, we have the natural identification of $C_t \cap (C_i - U_i) \times D$ with $(C_i - U_i)$ and thus $A_1, \ldots, A_{g_1+g_2}, B_1, \ldots B_{g_1+g_2}$ define a standard basis on each C_t. The following theorem is the core argument in our construction.

Theorem 5.2.1. *[2, p. 38] Up to replacing D by a neighborhood D_ϵ of 0, there exist $g_1 + g_2$ linearly independent holomorphic 2-forms $u_1, \ldots, u_{g_1+g_2}$ on $\mathcal{X}$ such that for every $t \in D_\epsilon$, the Poincaré residues of $\frac{u_i}{\pi_{\mathcal{X}} - t}$ along C_t form a normalized basis of holomorphic 1-forms with respect to $A_1, \ldots, A_{g_1+g_2}, B_1, \ldots B_{g_1+g_2}$. Moreover, the resulting Riemann matrix $\tau(t)$ is holomorphic with respect to t.*

We define the Poincaré residue of a top degree meromorphic form α on a complex manifold X with a pole being a smooth divisor D. In other words D is a submanifold of X of codimension one and α has a pole of order one along D. Let $x \in D$, $U \subset X$ a neighborhood of x and $z : U \to \mathbb{C}$ be a holomorphic function locally defining D with dz not vanishing along D. We can then locally write α on U as

$$\alpha = \frac{dz}{z} \wedge \beta + \gamma$$

where β and γ are holomorphic forms on U. The restriction of β to $D \cap U$ does not depend on the choices made and defines a global holomorphic form on D. We denote it by $\mathrm{res}_D(\alpha)$. In the case where the divisor of α is a sum of two smooth divisors D_1, D_2 intersecting transversely, we can analogously define $\mathrm{res}_{D_1}(\alpha)$ and $\mathrm{res}_{D_2}(\alpha)$ which will be meromorphic forms on D_1 and D_2 with at most simple poles along $D_1 \cap D_2$. See [4, p. 171] for a more thorough discussion on Poincaré residues.

In the theorem above, for $t = 0$, we get meromorphic forms on C_1 and C_2 with perhaps simple poles at p_1 and p_2. Since a meromorphic form on a Riemann surface cannot have one simple pole, we conclude that the resulting residues are in fact holomorphic.

We can now construct the family of Jacobians. Let

$$\mathcal{J} = \mathbb{C}^{g_1+g_2} \times D_\epsilon / \mathbb{Z}^{g_1+g_2} \times \mathbb{Z}^{g_1+g_2}$$

where the action of $\mathbb{Z}^{g_1+g_2} \times \mathbb{Z}^{g_1+g_2}$ on $\mathbb{C}^{g_1+g_2} \times D_\epsilon$ is given by

$$(\lambda, \gamma).(x, t) = (x + \lambda + \tau(t)\gamma, t).$$

Clearly the action is holomorphic, free and properly discontinuous and as such defines a complex manifold $\mathcal{J}$. Since the action preserves the t coordinate, we have the map

$$\pi_{\mathcal{J}} : \mathcal{J} \to D_\epsilon.$$

Clearly each fiber of $\pi_{\mathcal{J}}$ is the Jacobian of the corresponding fiber of $\pi_{\mathcal{X}}$.

We now construct the family

$$\pi_{\mathcal{J}_3(\mathcal{J})} : \mathcal{J}_3(\mathcal{J}) \to D_\epsilon$$

of the second intermediate co-Jacobians of fibers of $\pi_{\mathcal{J}}$. Let us digress for a moment in order to write down "explicitly" the second intermediate co-Jacobian of a torus

$$X = \mathbb{C}^g / \Lambda_\tau$$

where τ is a $g \times g$ matrix with complex coefficients such that $\Im\tau$ is positive definite and Λ_τ is the lattice spanned by the column vectors of the matrix (I, τ). We have that

$$J_3(X) = (H^{2,1}(X) \oplus H^{3,0}(X))^* / H_3(X, \mathbb{Z}).$$

We have that $H^{i,j}(X) \cong \mathcal{H}^{i,j}(X)$ where $\mathcal{H}^{i,j}(X)$ is the space of harmonic (i,j) forms on X. Now if α is a harmonic (i,j) form on X, we can lift it to a periodic harmonic form on $\mathbb{C}^g$. Any periodic harmonic form on $\mathbb{C}^g$ has constant coefficients. We therefore have that $H^{i,j}(X)$ is spanned by the forms $dz_{k_1} \wedge \cdots \wedge dz_{k_i} \wedge \overline{dz_{l_1}} \wedge \cdots \wedge \overline{dz_{l_j}}$ where z_i for $1 \le i \le g$ are the holomorphic basis of $\mathbb{C}^g$. In particular this gives an isomorphism

$$(H^{2,1}(X) \oplus H^{3,0}(X))^* \cong \mathbb{C}^{\binom{g}{2}\cdot\binom{g}{1}} \oplus \mathbb{C}^{\binom{g}{3}}.$$

Now as for the lattice $H_3(X, \mathbb{Z}) \subset (H^{2,1}(X) \oplus H^{3,0}(X))^*$, we have

$$H_3(X, \mathbb{Z}) \cong \bigwedge^3 \Lambda_\tau.$$

For $\lambda_1, \lambda_2, \lambda_3 \in \Lambda_\tau$, we have that the value of $\lambda_1 \wedge \lambda_2 \wedge \lambda_3$ on $\alpha \in H^{2,1}(X) \oplus H^{3,0}(X)$ when viewed as a constant form on $\mathbb{C}^g$ is the integral of α over the parallelepiped spanned by $\lambda_1, \lambda_2, \lambda_3$. This defines a lattice in $\mathbb{C}^{\binom{g}{1}\cdot\binom{g}{1}} \oplus \mathbb{C}^{\binom{g}{3}}$ under the isomorphism constructed above which we denote by $\Lambda_{\rho(\tau)}$. Important to note is that $\Lambda_{\rho(\tau)}$ varies holomorphically when τ does. In our case, we have the complex manifold

$$\mathcal{J}_3(\mathcal{J}) = \mathbb{C}^{\binom{g}{1}\cdot\binom{g}{1}} \oplus \mathbb{C}^{\binom{g}{3}} \times D_\epsilon / \Lambda_{\rho(\tau(t))},$$

which defines the desired family.

Let $q : D_\epsilon \to \mathcal{X}$ be a holomorphic section where by now, by $\mathcal{X}$ we mean the restriction of the family to D_ϵ. We can define the map

$$a_q : \mathcal{X} \to \mathcal{J}$$

which sends a point $p \in \mathcal{X}$ lying above $t \in D_\epsilon$ to

$$\Phi_1^{C_t}([p] - [q(t)]) \in J(C_t) \subset \mathcal{J}.$$

Theorem 5.2.2. *The map a_q is holomorphic.*

Proof. Let $s \in \mathcal{X}$ be the singular point of the fiber C_0. We have that $q(0) \neq s$. To see this, let $q(t) = (q_1(t), q_2(t)) \in D^2 \cong S$ where $q_1, q_2 \in \mathbb{C}[[t]]$ such that $q_1(t)q_2(t) = t$. If $q(0) = s$, we must have that q_1 and q_2 do not have constant factors and thus we cannot have $q_1(t)q_2(t) = t$.

Choose a point $p' \in \mathcal{X}$ lying over $t' \in D_\epsilon$ such that $p' \neq s$. We will show that a_q is holomorphic at p'. Consider a small neighborhood $U \subset \mathcal{X}$ of some path between p' and $q(t')$. For some neighborhood V of p', we can define a map $a_q^{'} : V \to \mathbb{C}^{g_1+g_2}$ given by

$$p \mapsto (\int_{q(t)}^{p} \mathrm{res}_{C_t} u_i)_{1 \leq i \leq g_1+g_2}$$

where p lies above t and the path of integration lies in $U \cap C_t$. Note that the result doesn't depend on the path chosen since all the paths between p and $q(t)$ in $U \cap C_t$ are homotopic and $\mathrm{res}_{C_t} u_i$ is closed. We have

$$a_q|_V = \Pi \circ (a_q', \pi_{\mathcal{X}})$$

where Π is the quotient map $\mathbb{C}^{g_1+g_2} \times D_\epsilon \to \mathcal{J}$. It thus suffices to show that $a_q^{'}$ is holomorphic. This will be done in several steps.

Step: 1 Assume first that p' and $q(t')$ lie in W_1 and U is holomorphically trivializable. Since W_1 is an open subset of $C_1 \times D$, we have coordinates (z, t) on U such that t coincides with $\pi_{\mathcal{X}}$. A holomorphic 2-form u_i can be written locally on U as

$$u_i = \phi(z, t) dt \wedge dz$$

where ϕ is a holomorphic function. It follows that $\mathrm{res}_{C_t \cap U} \frac{u_i}{\pi_{\mathcal{X}} - t} = \phi(z, t)dz$. The i-th coordinate of the map a_q' then has the following form

$$(z, t) \mapsto \int_{q(t)}^{z} \phi(z, t) dz$$

where $q(t)$ is holomorphic. This can be seen as an integral in $\mathbb{C}$ with holomorphically varying form and the base point. Clearly this map is holomorphic and thus so is a_q' at p'. The same is true if both points lie in W_2.

Step: 2 Assume now that $p' \in W_1 \cap S$, $t' = 0$ and $q(t') \in W_2 \cap S$. Consider any point $p'' \in V$ lying over $t'' \in D_\epsilon$ where $t'' \neq 0$. We have that $p'', q(t'') \in W_1$ since $S - C_0 \subset W_1$. It is moreover clear that p'' and $q(t'')$ lie in some holomorphically trivializable set. By step 1 we conclude that a_q is holomorphic on $V - C_0$. By Riemann's theorem on removable singularities, it suffices to show that a_q is continuous on $V \cap C_0$. Let x, y be coordinates on S given by the following isomorphism of D^2 and S,

$$(x, y) \to (x, y, xy) \in S \subset D^3.$$

Let

$$u_i = \phi(x, y) dx \wedge dy.$$

In order to calculate the residue of $\frac{u_i}{xy}$ on the set $x = 0$, we have to express $\frac{u_i}{xy}$ as

$$\frac{u_i}{xy} = \frac{dx}{x} \wedge \beta + \gamma$$

where β and γ could have simple poles along the set $y = 0$. We noted earlier that the residue of $\frac{u_i}{xy}$ along the set $x = 0$ is in fact holomorphic. In other words β is holomorphic and we have

$$\frac{u_i}{xy} = \frac{dx}{x} \wedge \beta + \frac{\gamma'}{y}$$

where β and γ' are holomorphic forms. We can thus express u_i as

$$u_i = y dx \wedge \beta + x\gamma'.$$

In particular, we can write

$$u_i = -x\phi_x(x,y)dx \wedge dy + y\phi_y(x,y)dx \wedge dy$$

for some holomorphic functions ϕ_x and ϕ_y. We thus have

$$\begin{aligned} \mathrm{res}_{x=0}u_i &= \phi_y dy, \\ \mathrm{res}_{y=0}u_i &= \phi_x dx. \end{aligned}$$

Since dx vanishes on the set $x = 0$ and dy vanished on the set $y = 0$ we can write

$$\begin{aligned} \mathrm{res}_{x=0}u_i &= \phi_y dy + \phi_x dx, \\ \mathrm{res}_{y=0}u_i &= \phi_y dy + \phi_x dx. \end{aligned}$$

We also have

$$\frac{u_i}{xy-t} = \frac{(ydx + xdy) \wedge (\phi_y dy + \phi_x dx)}{xy-t} = \frac{d(xy-t)}{xy-t} \wedge (\phi_y dy + \phi_x dx),$$

and therefore

$$\mathrm{res}_{xy=t}u_i = \phi_y dy + \phi_x dx.$$

We thus have that the i-th coordinate of the map a'_q has the following form

$$p \mapsto \int_{q(t)}^{p} \phi_y dy + \phi_x dx$$

where the path of integration is taken within the set $xy = t$. We can choose a path for every $p \in V$ such that they vary continuously with p. We thus get that a'_q is continuous everywhere on V.

Step: 3 Consider now the case when $p' \in W_1$ and $q(t') \in W_2$. By shrinking U and V if necessary, we can find a cover $(U_i)_{1 \leq i \leq n}$ of U by finitely many holomorphically trivial neighborhoods satisfying the following properties:
1) $q(t') \in U_1, p' \in U_n$,
2) $U_i \cap U_j = \emptyset$ if $i \neq j \pm 1$ for i and j distinct,
3) In case $t' = 0$, and so U contains the singular point s of C_0, we require that s belongs to only one open set, say U_l,
4) Each open set U_i belongs to either W_1 or W_2, except for the set U_l in the case $t' = 0$ where we then require that $U_l \subset S$.
5) There is some neighborhood $D'_\epsilon \subset D_\epsilon$ of t' such that $\pi_{\mathcal{X}}(V) = \pi_{\mathcal{X}}(U_i \cap U_{i+1}) = D'_\epsilon$ for $i = 1, \ldots, n-1$.
It is easy to see that such a cover exists. We now choose some holomorphic sections

$$q_i : D'_\epsilon \to U_i \cap U_{i+1}.$$

For any $p \in V$ lying over t, we have

$$a'_q(p) = a'_q(q_1(t)) + \sum_{i=1}^{n-2} a'_{q_i}(q_{i+1}(t)) + a'_{q_n}(p).$$

In case $t' \neq 0$, by step 1 we know that each term in this sum is holomorphic since it involves calculations within some trivializable set in W_1 or W_2. In case $t' = 0$, we have that $a'_{q_{l-1}}(q_l(t))$ is holomorphic by step 2.

So far we have shown that the map a_q is holomorphic outside of s. By Hartog's extension theorem, a continuous function cannot fail to be holomorphic on a set of codimension more than one. It thus follows that a_q is holomorphic everywhere.

□

We have a flat analytic cycle $a_q(\mathcal{X}) \subset \mathcal{J}$. Let

$$C_q(\mathcal{X}) := a_q(\mathcal{X}) - [-1]^* a_q(\mathcal{X}),$$

where $[-1]$ is the map on $\mathcal{J}$ that acts by inversion on each fiber of $\pi_{\mathcal{J}}$, be the Ceresa cycle of $\mathcal{X}$. We have that $C_q(\mathcal{X}) \cap C_t$ is the Ceresa cycle of C_t and as such is homologous to 0. By the theorem of Griffiths [3], since $C_q(\mathcal{X})$ is flat, we have that the map

$$c_q(\mathcal{X}) : D_\epsilon \to \mathcal{J}_3(\mathcal{J}_1)$$

which sends $t \in D_\epsilon$ to the image of $C_q(\mathcal{X}) \cap C_t$ under the co-Abel-Jacobi map $\Phi_2^{C_t}$ in $\pi^{-1}_{\mathcal{J}_3(\mathcal{J})}(t)$ is holomorphic. We have that $c_q(t)$ is the Ceresa element of C_t.

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